Exposition on affine and elliptic root systems and elliptic Lie algebras

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Abstract

This is an exposition in order to give an explicit way to understand (1) a non-topological proof for an existence of a base of an affine root system, (2) a Serre-type definition of an elliptic Lie algebra with rank ≥ 2, and (3) the isotropic root multiplicities obtained from a viewpoint of the Saito-marking lines.

1 Introduction

In 1985, K. Saito [16] introduced the notion of an *n-extended affine root system*. If $n = 0$ (respectively, $n = 1$), it is an irreducible finite root system (respectively, an affine root system). In [16], he also intensively studied 2-extended affine root systems, which are now called *elliptic root systems* (see [17]). Since then, various attempts have been made to construct Lie algebras whose non-isotropic roots form those root systems. Among them are *toroidal Lie algebras* [15], *extended affine Lie algebras* [1], and *toral type extended affine Lie algebras* [15], [21]. See [18, Introduction] for the history.

In 2000, K. Saito and D. Yoshii [18] constructed certain Lie algebras by using the Borcherds lattice vertex algebras, called them *simply-laced elliptic Lie algebras* and showed that they are isomorphic to ADE-type (2-variable) toroidal Lie algebras of rank ≥ 2. They also gave two other definitions for their Lie algebras. One uses generators and relations. The other uses (affine-type) Heisenberg Lie algebras; this was generalized by D. Yoshii [20] in order to define Lie algebras associated with the reduced elliptic root systems, and he called them *elliptic Lie algebras*. In 2004, the second author [19] gave defining relations of the elliptic Lie algebras of rank ≥ 2.

The aim of this paper is to give an exposition in order to give an explicit way to understand (1) a non-topological proof for the existence of a base of an affine root system (Theorem 3.1, originally given in [13]), (2) a Serre-type definition of an elliptic Lie algebra $\mathfrak{g}$ with rank ≥ 2 (Definition 5.1, originally given in [19]) and the fact that the non-isotropic roots form the corresponding elliptic root system and their multiplicities are one (Theorem 5.1, originally given in [19]), and (3)

a list of the multiplicities of the isotropic roots of $\mathfrak{g}$, proved from a viewpoint of the Saito-marking lines (Theorem 6.1, new result).

As for (2), we point out that our defining relations are closely related to defining relations, called *Drinfeld realization*, of the quantum affine algebras due to V.G. Drinfeld [7, Theorems 3 and 4]. Recently the same authors have written a paper [5], motivated by [22], giving a finite number of defining relations of the universal coverings of some Lie tori.

We hope that the material presented here regarding affine root systems, in particular the existence of a base, would give another point of view to readers interested in the subject, specially to those reading the book [14] by I.G. MacDonald. (Incidentally, in order to read [14], we also hope that the paper [8] would also be helpful in being familiar with Coxeter groups, especially the Matsumoto theorem.)

2 Preliminary

In this section, we mention elemental properties of (Saito's) extended affine root systems, especially (2.5).

2.1 Basic notation and terminology

As usual, we let $\mathbb{Z}$ denote the ring of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers. For a set S, let $|S|$ denote the cardinal number of S. If S is a subset of $\mathbb{R}$, we let $S^\times := \{s \in S | s \neq 0\}$, $S_+ := \{s \in S | s \geq 0\}$, and $S_- := \{s \in S | s \leq 0\}$.

For a unital subring X of $\mathbb{C}$, an X-module M, a subset Y of X, subsets S and S' of M, $x \in X$ and $m \in M$, we let $S + S' := \{m + m' \in M | m \in S, m' \in S'\}$, $m + S := \{m\} + S$, $YS := \{y_1 s_1 + \cdots + y_r s_r | r \in \mathbb{N},\, y_i \in Y,\, s_i \in S\,(1 \leq i \leq r)\}$, $Ym := Y\{m\}$, $xS := \{x\}S$ and $-S := (-1)S$; we understand $S + \emptyset = \emptyset$, $\emptyset S = \emptyset$ and $Y\emptyset = \emptyset$.

Throughout this paper, for any $\mathbb{F}$-linear space $\mathcal{V}$ with a symmetric bilinear form $(\,,\,) : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, we set $\mathcal{V}^0 := \{v \in \mathcal{V} | (v, v) = 0\}$ and $\mathcal{V}^\times := \mathcal{V} \setminus \mathcal{V}^0$; for each $v \in \mathcal{V}^\times$, we set $v^\vee := \frac{2v}{(v,v)}$ and define $s_v \in \mathrm{GL}(\mathcal{V})$ by $s_v(z) = z - (v^\vee, z)v$ $(z \in \mathcal{V})$; for any non-empty subset S of $\mathcal{V}^\times$, we denote by W_S the subgroup of $\mathrm{GL}(\mathcal{V})$ generated by $\{s_v | v \in S\}$, i.e.,

$$W_S := \langle s_v | v \in S \rangle, \tag{2.1}$$

and moreover, let $W_S \cdot S' := \{w(z') \in \mathcal{V} | w \in W_S, z' \in S'\}$, $W_S \cdot z := W_S \cdot \{z\}$ for a subset S' of $\mathcal{V}$ and $z \in \mathcal{V}$, and say that a subset S of $\mathcal{V}^\times$ is *connected* if there exists no non-empty proper subset S' of S with $(S', S \setminus S') = \{0\}$. For a subset $\mathcal{V}'$ of $\mathcal{V}$, let $(\mathcal{V}')^0 := \mathcal{V}' \cap \mathcal{V}^0$, and $(\mathcal{V}')^\times := \mathcal{V}' \cap \mathcal{V}^\times$. We call an element of $\mathcal{V}^0$ *isotropic*.

In this paper, we always let

$$\pi : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0 \tag{2.2}$$

denote the canonical map.

2.2 Extended affine root systems

Definition 2.1. Let $l \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Let $\mathcal{V}$ be an $(l+n)$-dimensional $\mathbb{R}$-linear space. Recall $\mathcal{V}^0$ and $\mathcal{V}^\times$ from Subsection 2.1. Assume that there exists a positive semi-definite symmetric bilinear form $(\, , \,) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that $\dim_{\mathbb{R}} \mathcal{V}^0 = n$. Let R be a subset of $\mathcal{V}$. Then R (or more precisely, $(R, \mathcal{V})$) is an (n-)*extended affine root system* if R satisfies the following axioms:

(AX1) $R \subset \mathcal{V}^\times$, $\mathcal{V} = \mathbb{R}R$.
(AX2) $\mathbb{Z}R$ is free as a $\mathbb{Z}$-module, and $\mathrm{rank}_{\mathbb{Z}}\mathbb{Z}R = n + l (= \dim_{\mathbb{R}} \mathcal{V})$.
(AX3) $(\alpha^\vee, \beta) \in \mathbb{Z}$ for α, $\beta \in R$.
(AX4) $s_\alpha(R) = R$ for all $\alpha \in R$.
(AX5) R is connected.

(see [16, (1.2) Definition 1 and (1.3) Note 2 iii)] and see [2] for an equivalence to [1, Definition 2.1].) Let $W = W_R$ (see (2.1)).

Let R be as in Definition 2.1. It is well-known that for all $\alpha \in R$,

$$\left\{ \begin{array}{l} R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}, \{\alpha, 2\alpha, -\alpha, -2\alpha\} \text{ or } \{\alpha, \frac{1}{2}\alpha, -\alpha, -\frac{1}{2}\alpha\}, \\ (\text{so } -R = R). \end{array} \right. \tag{2.3}$$

We call R *reduced* (resp. *non-reduced*) if $R \cap 2R = \emptyset$ (resp. $R \cap 2R \neq \emptyset$).

We say that two extended affine root systems $(R, \mathcal{V})$ and $(R', \mathcal{V}')$ are *isomorphic* if there exist an $\mathbb{R}$-linear bijective map $f : \mathcal{V} \to \mathcal{V}'$ and $c \in \mathbb{R}$ with $c > 0$ such that $f(R) = R'$ and $(f(v), f(w)) = c(v, w)$ for v, $w \in \mathcal{V}$.

(2.4) We call this f a *root system isomomorphism*.

Let R, l and n be as above.

By [12, Theorem 5 of Chapter XV], since $\mathbb{Z}R/(\mathbb{Z}R)^0$ is torsion free, (AX1-5) imply that there exists an $\mathbb{R}$-basis $\{x_1, \ldots, x_{l+n}\}$ of $\mathcal{V}$ such that $\{x_{l+1}, \ldots, x_{l+n}\}$ is an $\mathbb{R}$-basis of $\mathcal{V}^0$, $\{x_1, \ldots, x_{l+n}\}$ is a $\mathbb{Z}$-basis of the (torsion) free $\mathbb{Z}$-module $\mathbb{Z}R$ and $\{x_{l+1}, \ldots, x_{l+n}\}$ is a $\mathbb{Z}$-basis of the (torsion) free $\mathbb{Z}$-module $(\mathbb{Z}R)^0$ (see Subsection 2.1 for notation), that is,

$$\left\{ \begin{array}{l} \mathcal{V} = \mathbb{R}R = \bigoplus_{i=1}^{l+n} \mathbb{R}x_i, \ \mathcal{V}^0 = \bigoplus_{j=l+1}^{l+n} \mathbb{R}x_j, \\ \mathbb{Z}R = \bigoplus_{i=1}^{l+n} \mathbb{Z}x_i, \ (\mathbb{Z}R)^0 = \bigoplus_{j=l+1}^{l+n} \mathbb{Z}x_j, \\ \dim_{\mathbb{R}} \mathcal{V} = \mathrm{rank}_{\mathbb{Z}}\mathbb{Z}R = n + l, \ \dim_{\mathbb{R}} \mathcal{V}^0 = \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}R)^0 = n. \end{array} \right. \tag{2.5}$$

Let $\{a_1, \ldots, a_n\}$ be a $\mathbb{Z}$-basis of $(\mathbb{Z}R)^0$. Then there exist $x_1, \ldots, x_l \in \mathbb{Z}R$ such that $\{x_1, \ldots, x_l, a_1, \ldots, a_n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}R$ as well as an $\mathbb{R}$-basis of $\mathcal{V} = \mathbb{R}R$ (see above). Let $1 \leq m \leq n$. Let $\pi' : \mathcal{V} \to \mathcal{V}/(\mathbb{R}a_m \oplus \cdots \oplus \mathbb{R}a_n)$ be the canonical map. Note that $\{\pi'(x_1), \ldots, \pi'(x_l), \pi'(a_1), \ldots, \pi'(a_{m-1})\}$ is an $\mathbb{X}$-basis of $\mathbb{X}\pi'(R)$ for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$. In particular, we see that

(2.6) if $y_1, \ldots, y_{l+m-1}$ are elements of $\mathbb{Z}R$ such that $\{\pi'(y_1), \ldots, \pi'(y_{l+m-1})\}$ is a $\mathbb{Z}$-base of the free $\mathbb{Z}$-module $\mathbb{Z}\pi'(R)$, then $\{y_1, \ldots, y_{l+m-1}, a_m, \ldots, a_n\}$ is an $\mathbb{X}$-basis of $\mathbb{X}R$ for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$.

(2.7) We call l the *rank* of R. We call n the *nullity* of R.

If $n = 0$, then R is an *irreducible finite root system* (see [16, (1.3) Example 1 i)]). If $n = 1$, then R is an *affine root system* (see [16, (1.3) Example 1 ii)]), see also Remark 2.1 below. If $n = 2$, then R is an *elliptic root system* (see [16, (1.3) Example 1 iii)], [17] and [18]).

Remark 2.1. Assume $n = 1$. Here we give a sketch of a proof of an equivalence between affine root systems in the senses of [13], [14, §1.2] and [16] (i.e. our Definition 2.1). Let F and E be as in [14, §1.2]. Let S be a subset of F, and assume S is an irreducible affine root system in the sense of [14, §1.2]. Identify $\mathcal{V}$ with F, that is, we regard $\mathcal{V}$ as an $l+1$-dimensional $\mathbb{R}$-linear space of affine-linear functions $f : E \to \mathbb{R}$. Clearly S satisfies (AX1) and (AX3-5). Let $\lambda \in \mathcal{V}^\times$. Let $\mu \in \mathcal{V}^\times$ be such that $c\mu \in \lambda + \mathcal{V}^0$ for some $c \in \mathbb{R}^\times$. Then $\lambda - c\mu$ is a constant function on E, that is, $(\lambda - c\mu)(E) = \{d_{\lambda - c\mu}\}$ for some $d_{\lambda - c\mu} \in \mathbb{R}$. We have $s_\mu s_\lambda(x) = x - (\lambda^\vee, x)(\lambda - c\mu)$ for $x \in \mathcal{V}$. Further, for $e \in E$, we have $s_\mu s_\lambda \cdot e = e + \frac{2d_{\lambda - c\mu}}{(\lambda,\lambda)} D\lambda$, see [14, §1.1] for $D\lambda$. Then by using an argument similar to [16, (1.16) Assertion 1], we can see that S satisfies (AX2). Let R be as in Definition 2.1. Let T be the subgroup of W generated by $\{s_\alpha s_{\alpha'} \mid \alpha, \alpha' \in R, \mathbb{R}^\times \pi(\alpha) = \mathbb{R}^\times \pi(\alpha')\}$. Then T is a normal abelian subgroup, and W/T can be identified with the finite Weyl group $W_{\pi(R)}$ (cf. [16, (1.3) Note 2 ii)]). Then R satisfies (AR 4) of [14, §1.2].

2.3 Base of an irreducible finite or affine root system

Assume that $n \in \{0, 1\}$ (cf. (2.7)). We call a subset Π of R formed by $(l+n)$-linearly independent elements a *base* if

(2.8) $$R = (R \cap \mathbb{Z}_+\Pi) \cup (R \cap \mathbb{Z}_-\Pi).$$

(For $n = 0$, see [9, Theorem 10.1]. For $n = 1$, see Theorem 3.1 in this paper (cf. MacDonald [13, (4.6)] (see also [16, (3.3) i)-iii)]))). If Π is a base of R, then, for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

(2.9) Π is an $\mathbb{X}$-basis of $\mathbb{X}R$, that is, $\mathbb{X}R = \bigoplus_{\alpha \in \Pi} \mathbb{X}\alpha$.

Assume that $n = 1$. Let $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$ be a base of R; we always assume α_0 is such that $\{\pi(\alpha_1), \dots, \pi(\alpha_l)\}$ is a base of $\pi(R)$ (see Theorem 3.1). Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

(2.10) $\delta(\Pi) \in \mathbb{N}\Pi$ and $\{\delta(\Pi)\}$ is a $\mathbb{Z}$-basis of $(\mathbb{Z}R)^0$, that is, $\mathbb{Z}\delta(\Pi) = (\mathbb{Z}R)^0$.

$\delta(\Pi)$ is unique by (2.5). By (2.6), for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

(2.11) $\{\alpha_1, \dots, \alpha_l, \delta(\Pi)\}$ is a $\mathbb{X}$-basis of $\mathbb{X}R$, that is, $\mathbb{X}R = (\bigoplus_{i=1}^{n} \mathbb{X}\alpha_i) \oplus \mathbb{X}\delta(\Pi)$.

The following lemma is well-known, e.g., see [9, Theorem 10.3, Lemmas 10.4 C,D, §12 Excercises 3].

Lemma 2.1. *Assume that $n = 0$ (cf. (2.7)). Let Π be a base of R (cf. (2.8)). Then we have the following*:

(1) $W_\Pi = W$ *and* $W \cdot \Pi = R \setminus 2R$. (*see* (2.1) *for* W_Π *and see Definition* 2.1 *for* $W = W_R$).

(2) $W \cdot \alpha = \{\beta \in R | (\alpha, \alpha) = (\beta, \beta)\}$ *for each* $\alpha \in R$.

(3) *For each* $\alpha \in R$, *there exists a unique* $\alpha_+ \in W \cdot \alpha$ *such that* $W \cdot \alpha \subset \alpha_+ + \mathbb{Z}_-\Pi$.

(4) *Let* $r = |\{(\alpha, \alpha) | \alpha \in R\}|$. *Then* $1 \leq r \leq 3$. *Moreover, if* $r = 3$, *then* $R \cap 2R = \{\beta \in R \,|\, (\beta, \beta) \geq (\alpha, \alpha)$ *for all* $\alpha \in R\}$.

Proof of (3). Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$. Then α_+ is the element $\sum_{i=1}^{l} m_i\alpha_i \in W \cdot \alpha$ ($m_i \in \mathbb{Z}$) for which $\sum_{i=1}^{l} m_i$ is maximal. Let $w \in W_\Pi$ and let $w = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced expression, that is, r is as small as possible. By [9, Corollary 10.2 C], we have $w.\alpha_+ = \alpha_+ - \sum_{j=1}^{r} (\alpha_j^\vee, \alpha_+) s_{\alpha_1} \cdots s_{\alpha_{j-1}}(\alpha_j) \in \alpha_+ + \mathbb{Z}_-\Pi$. □

For R and Π of Lemma 2.1, we let

(2.12) $$\Theta(R, \Pi) := \{\alpha_+ \in R | \alpha \in R\}.$$

By checking directly (and using [9, §12 Table 2]), we have

(2.13) $$(\mu, \nu) > 0 \text{ for } \mu, \nu \in \Theta(R, \Pi).$$

2.4 Notation $S_{\rm sh}$, $S_{\rm lg}$, $S_{\rm ex}$

Let R be an (n-)extended affine root system (see Definition 2.1). Define the subsets $R_{\rm sh}$, $R_{\rm lg}$ and $R_{\rm ex}$ of R by

$$R_{\rm sh} := \{\alpha \in R \,|\, (\alpha, \alpha) \leq (\beta, \beta) \text{ for all } \beta \in R\},$$

$R_{\mathrm{ex}} := R \cap \pi^{-1}(2\pi(R_{\mathrm{sh}}))$ and $R_{\mathrm{lg}} := R \setminus (R_{\mathrm{sh}} \cup R_{\mathrm{ex}})$ (see (2.2) for π). Then we have

$$R = R_{\mathrm{sh}} \cup R_{\mathrm{lg}} \cup R_{\mathrm{ex}} \text{ (disjoint union).} \tag{2.14}$$

For a subset S of R, let

$$S_{\mathrm{sh}} := S \cap R_{\mathrm{sh}},\ S_{\mathrm{lg}} := S \cap R_{\mathrm{lg}},\ S_{\mathrm{ex}} := S \cap R_{\mathrm{ex}}. \tag{2.15}$$

3 A non-topological proof for the existence of a base of an affine root system

In this section we assume R is an affine root system, that is, we assume $n = 1$ (see (2.7)).

3.1 The existence of a base of an affine root system

The following theorem seems to be well-known (see [13]), but we state and prove it for later use. The proof in [13] uses topological terminology. Our proof seems to be the first one without using topology. Besides we need a technically written statement of the following theorem for application.

Theorem 3.1. (*cf.* [13]) *Let $\delta' \in \mathcal{V}^0 \setminus \{0\}$ be such that $\mathbb{Z}\delta' = (\mathbb{Z}R)^0$ (cf. (2.5)). Let $\Pi' = \{\alpha_1, \ldots, \alpha_l\}$ be a subset of R with $|\Pi'| = l$ such that $\pi(\Pi')$ is a base of the irreducible finite root system $(\pi(R), \mathcal{V}/\mathbb{R}\delta')$ (cf. (2.8) and (2.2)). (So $\mathbb{Z}R = \mathbb{Z}\delta' \oplus \mathbb{Z}\Pi'$ (cf. (2.6)).) Then there exists a unique*

$$\alpha_0 = \alpha_0(R, \Pi', \delta') \in R \tag{3.1}$$

such that $\{\alpha_0\} \cup \Pi'$ is a base of R and $\alpha_0 \in \mathbb{N}\delta' \oplus \mathbb{Z}\Pi'$. Moreover $\alpha_0 = \delta' - \theta$ for some $\theta \in \mathbb{N}\Pi'$ with $\pi(\theta) \in \Theta(\pi(R), \pi(\Pi'))$ (see (2.12)). In particular, $[(\alpha_i^\vee, \alpha_j)]_{0 \leq i,j \leq l}$ is a generalized Cartan matrix of affine-type in the sense of [10, §4.3 and Proposition 4.7]. Further, letting $\Pi_1 = \{\alpha_0\} \cup \Pi'$, for any base Π_2 of R we have $\Pi_2 = \epsilon w(\Pi_1)$ for some $\epsilon \in \{1, -1\}$ and $w \in W_{\Pi_1}$. In particular,

$$R \setminus 2R = W_{\Pi_1} \cdot \Pi_1 \text{ and } W = W_{\Pi_1}. \tag{3.2}$$

Proof. (*Strategy.* We use a linear map $f : \mathcal{V} \to \mathbb{R}$ (i.e., $f \in \mathcal{V}^*$) such that $f(\alpha_i) = 1$ $(1 \leq i \leq l)$ and $f(\delta')$ is sufficiently large (see (3.6)). Let Π^f be the subset of R formed by the elements $\beta \in R$ satisfying the condition that $f(\beta) > 0$ and β is not expressed as the summation of more than one elements β' of R with $f(\beta') > 0$ (see (3.8)). We show that Π^f is a base of R satisfying the properties of the statement. It is easy to see that $\Pi' \subset \Pi^f$ and $R = (R \cap \mathbb{Z}_+\Pi^f) \cup (R \cap \mathbb{Z}_-\Pi^f))$. We show $|\Pi^f| = l + 1$ by using (2.13).)

We proceed with the proof of the theorem in the following steps.

Step 1 (*Definition of* f). Notice that for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$,

$$\mathbb{X}R = \mathbb{X}\delta' \oplus (\oplus_{i=1}^{l} \mathbb{X}\alpha_i) \tag{3.3}$$

(see (2.6)). We may assume that $(\alpha_i, \alpha_i) \le (\alpha_{i+1}, \alpha_{i+1})$ for $1 \le i \le l-1$. Also since $\pi(\Pi')$ is a base of $\pi(R)$, if $l \ge 2$, we may assume α_1 is such that there exists a unique $j \in \{2, \dots, l\}$ such that $(\alpha_1, \alpha_j) \ne 0$. Let

$$R' := \begin{cases} W_{\Pi'} \cdot (\Pi' \cup \{2\alpha_1\}) & \text{if } l = 1, \\ W_{\Pi'} \cdot (\Pi' \cup \{2\alpha_1\}) & \text{if } l \ge 2 \text{ and } 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2), \\ W_{\Pi'} \cdot \Pi' & \text{otherwise.} \end{cases} \tag{3.4}$$

Using [9, Theorem 10.3 (c) (and §12 Exercise 3)], we can see that $W_{\Pi'} \cdot \Pi'$ and R' are irreducible finite root systems with the base Π'. If $\pi(R)$ is reduced, then $\pi(R) = \pi(W_{\Pi'} \cdot \Pi')$. If $\pi(R)$ is not reduced, then $\pi(R) = \pi(R')$. In particular, we have

$$R \subset R' + \mathbb{Z}\delta'. \tag{3.5}$$

(see also (3.3)).

Define $f \in \mathcal{V}^*$ by

$$f(\alpha_i) = 1 \ (1 \le i \le l) \quad \text{and} \quad f(\delta') = 3M, \tag{3.6}$$

where $M := \max\{|f(\gamma)| | \gamma \in R'\}$ (notice $|R'| < \infty$). It follows from (3.5) that $f(\beta) \ne 0$ for $\beta \in R$.

Step 2 (*Definition of* Π^f). Let $R^{f,+} := \{\beta \in R | f(\beta) > 0\}$. By (3.6), we have

$$R^{f,+} = R \cap ((R' \cap \mathbb{Z}_+\Pi') \cup (\cup_{m=1}^{\infty} (m\delta' + R'))). \tag{3.7}$$

Let Π^f be a subset of R formed by the elements $\beta \in R^{f,+}$ satisfying the condition that there exist no $\beta_1, \dots, \beta_r \in R^{f,+}$ with $r \ge 2$ such that $\beta = \beta_1 + \cdots + \beta_r$; namely,

$$\Pi^f := R^{f,+} \setminus (\bigcup_{r=2}^{\infty} \{\sum_{i=1}^{r} \beta_i | \beta_i \in R^{f,+}\}). \tag{3.8}$$

By (3.7), we have

$$\Pi' \subset \Pi^f. \tag{3.9}$$

Notice $\mathbb{Z}\Pi' \ne \mathbb{Z}R$ (by (3.3)). Then we have

$$\mathbb{Z}\Pi^f = \mathbb{Z}R, \ R = (R \cap \mathbb{Z}_+\Pi^f) \cup (R \cap \mathbb{Z}_-\Pi^f) \text{ and } |\Pi^f| \ge |\Pi'| + 1. \tag{3.10}$$

(As mentioned in our strategy, we show that Π^f is a base of R.)

Step 3 (*If* $\beta\in\Pi^f/\Pi'$, *then we have* $\pi(\beta)\in\Theta(\pi(R),\pi(\Pi'))$ (*for* $\Theta(\pi(R),\pi(\Pi'))$, *see* (2.12))). Let $\beta\in\Pi^f/\Pi'$ (see also (3.9)-(3.10)). We show that β is expressed as

$$\beta=m\delta'-\theta \tag{3.11}$$

for some $m\in\mathbb{N}$ and some θ with

$$\theta\in\Theta(R',\Pi') \tag{3.12}$$

(see (2.12) for $\Theta(R',\Pi')$). By (3.7), since $\Pi^f\subset R^{f,+}$, we have

$$\beta=m\delta'+\mu \tag{3.13}$$

for some $m\in\mathbb{N}$ and $\mu\in R'$. Let $\theta\in\Theta(R',\Pi')\cap W_{\Pi'}\cdot\mu$, where we recall from Lemma 2.1 (2)-(3) that $|\Theta(R',\Pi')\cap W_{\Pi'}\cdot\mu|=1$. Notice $\{\mu,-\mu,\theta,-\theta\}\subset W_{\Pi'}\cdot\mu$ (cf. Lemma 2.1 (2)). Then $m\delta'-\theta\in R$ since $m\delta'-\theta\in m\delta'+W_{\Pi'}\cdot\mu=W_{\Pi'}\cdot(m\delta'+\mu)=W_{\Pi'}\cdot\beta\subset R$. By Lemma 2.1 (3), we have $\theta+\mu=\theta-(-\mu)\in\mathbb{Z}_+\Pi'$. Since $m\delta'-\theta\in R^{f,+}$ (cf. (3.7)), $\beta=(m\delta'-\theta)+(\theta+\mu)$ and $\beta\in\Pi^f$, we have $\theta+\mu=0$ and (3.11), as desired.

Step 4 ($|\Pi^f|=l+1$). We show

$$|\Pi^f\setminus\Pi'|=1,\ \text{i.e.,}\ |\Pi^f|=l+1 \tag{3.14}$$

(see also (3.9)-(3.10)).

Assume $|\Pi^f\setminus\Pi'|>1$. Let β_1, $\beta_2\in\Pi^f\setminus\Pi'$ and assume $\beta_1\neq\beta_2$. Assume $(\beta_1,\beta_1)\leq(\beta_2,\beta_2)$. Then, by (2.13) and (3.11)-(3.12), we see that

$$(\beta_2^\vee,\beta_1)=\begin{cases}1 & \text{if } \pi(\beta_1)\neq\pi(\beta_2),\\ 2 & \text{if } \pi(\beta_1)=\pi(\beta_2).\end{cases}$$

Assume $(\beta_2^\vee,\beta_1)=1$. Then, since $\pm(\beta_1-\beta_2)=s_{\beta_2}(\pm\beta_1)\in R$, we have $\beta_1-\beta_2\in R^{f,+}$ or $\beta_2-\beta_1\in R^{f,+}$. This contradicts the fact β_1, $\beta_2\in\Pi^f$ since $\beta_1=\beta_2+(\beta_1-\beta_2)$ and $\beta_2=\beta_1+(\beta_2-\beta_1)$. Assume $(\beta_2^\vee,\beta_1)=2$, so $\pi(\beta_1)=\pi(\beta_2)$. By (3.11), there exist n_1, $n_2\in\mathbb{N}$ and $\theta\in\Theta(R',\Pi')$ such that

$$\beta_i=n_i\delta'-\theta\quad(i\in\{1,2\})$$

(so $\beta_2-\beta_1=(n_2-n_1)\delta'$). Assume $n_1<n_2$. Notice that for $i\in\{1,2\}$ and $r\in\mathbb{Z}$,

$$\begin{aligned} R &\ni (s_{\beta_2}s_{\beta_1})^r(\beta_i)\quad\text{(by (AX4))}\\ &= (n_i+2r(n_2-n_1))\delta'-\theta\\ &= \begin{cases}(n_2+(2r-1)(n_2-n_1))\delta'-\theta & \text{if } i=1,\\ (n_2+2r(n_2-n_1))\delta'-\theta & \text{if } i=2.\end{cases}\end{aligned} \tag{3.15}$$

Hence

$$(3.16)\qquad (n_2+r(n_2-n_1))\delta'-\theta\in R \quad \text{for all } r\in\mathbb{Z}.$$

Let $n_3\in\mathbb{Z}_+$ and $t\in\mathbb{N}$ be such that $0\le n_3<n_2-n_1$ and $n_2=t(n_2-n_1)+n_3$. Assume $n_3=0$. By (3.16), $\{-\theta,(n_2-n_1)\delta'-\theta\}\subset R$. Hence, by (3.7) (and (2.3)), $\{\theta,(n_2-n_1)\delta'-\theta\}\subset R^{f,+}$. Notice $t\ge 2$ (since $0<n_1<n_2$ and $n_3=0$). Since $\beta_2=t((n_2-n_1)\delta'-\theta)+(t-1)\theta$, we have $\beta_2\notin\Pi^f$, contradiction. Assume $n_3>0$. Notice $2n_3<n_2$ (since $2n_3<(n_2-n_1)+n_3\le t(n_2-n_1)+n_3=n_2$). Let $\beta_3=n_3\delta'-\theta$. By (3.16), $\beta_3\in R$. By (3.7), $\beta_3\in R^{f,+}$. Notice $\beta_2-2\beta_3=s_{\beta_3}(\beta_2)\in R$ (by (AX4)). Then by (3.7), we have

$$\beta_2-2\beta_3=(n_2-2n_3)\delta'+\theta\in R^{f,+}.$$

Since $\beta_2=(\beta_2-2\beta_3)+2\beta_3$, we have $\beta_2\notin\Pi^f$, contradiction. Hence $|\Pi^f|=l+1$, as desired.

Step 5 (Π^f *is a base with* $\alpha_0=\delta'-\theta$). Let α_0 be $\beta=m\delta'-\theta$ of (3.11). Then $\Pi^f=\Pi'\cup\{\alpha_0\}$, where we notice (3.9) and (3.14). It is clear that the elements of Π^f are linearly independent (cf. (3.3)). Hence, by (3.10), Π^f is a base of R (cf. (2.8)). Since $\mathbb{Z}\Pi'\oplus\mathbb{Z}\delta'=\mathbb{Z}\Pi'\oplus\mathbb{Z}\alpha_0$ (by (3.3) and (3.10)), we have $m=1$.

Step 6 (*The last claim holds*). Let $\Pi_1=\Pi'\cup\{\alpha_0\}$. Let Π_2 be a base of R. Define $h\in V^*$ by $h(\beta):=1$ ($\beta\in\Pi_2$). Then $h(R)\subset\mathbb{Z}\setminus\{0\}$. By the same formula as in (3.15), we have $|\{(s_\theta s_{\alpha_0})^r(\alpha_0)\in R|r\in\mathbb{Z}\}|=\infty$ (notice that $(s_\theta s_{\alpha_0})^r(\alpha_0)\in R$ (by (AX4)) since $s_\theta=s_{\frac{1}{2}\theta}$ and $\theta\in R\cup 2R$ (see (3.12) and (3.4))). Hence $|R|=\infty$, which implies $|h(R)|=\infty$. Hence, by (3.5), since $|R'|<\infty$ (R' is an irreducible finite root system), we have $h(\delta')\ne 0$. We may assume

$$(3.17)\qquad h(\delta')>0$$

(otherwise, we replace Π_2 with $-\Pi_2$). Let

$$\begin{aligned} m(\Pi_1,\Pi_2) &:= |(R\cap\mathbb{Z}_+\Pi_1\cap\mathbb{Z}_-\Pi_2)\setminus 2R| \\ &= |\{\beta\in(R\cap\mathbb{Z}_+\Pi_1)\setminus 2R\,|\,h(\beta)<0\}|. \end{aligned}$$

Since $\alpha_0=\delta'-\theta$, we have $R\cap\mathbb{Z}_+\Pi_1\subset R'+\mathbb{Z}_+\delta'$ (cf. (3.5)). Hence, since $|R'|<\infty$, by (3.17), we have $m(\Pi_1,\Pi_2)<\infty$.

We use induction on $m(\Pi_1,\Pi_2)$; if $m(\Pi_1,\Pi_2)=0$, then, by (2.8), $R\cap\mathbb{Z}_+\Pi_1=R\cap\mathbb{Z}_+\Pi_2$, so $\Pi_1=\Pi_2$. Assume $m(\Pi_1,\Pi_2)>0$. Then there exists $\alpha\in\Pi_1$ such that $\alpha\in\mathbb{Z}_-\Pi_2$ (notice that $R\subset\mathbb{Z}_-\Pi_2\cup\mathbb{Z}_+\Pi_2$). By (2.8) (and (2.3)), we see

$$(3.18)\qquad s_\alpha((R\cap\mathbb{Z}_+\Pi_1)\setminus 2R)=\{-\alpha\}\cup(((R\cap\mathbb{Z}_+\Pi_1)\setminus 2R)\setminus\{\alpha\}).$$

Then we have

$$
\begin{aligned}
& m(\Pi_1, s_\alpha(\Pi_2)) \\
& \quad = \quad |(R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R| \\
& \quad = \quad |s_\alpha((R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R)| \\
& \quad = \quad |(s_\alpha(R \cap \mathbb{Z}_+\Pi_1) \cap \mathbb{Z}_-\Pi_2) \setminus 2R| \\
& \quad = \quad m(\Pi_1, \Pi_2) - 1 \quad \text{(by (3.18) since } s_\alpha(\alpha) = -\alpha \notin \mathbb{Z}_-\Pi_2).
\end{aligned}
$$

Then, by the induction, we see that there exists $w \in W_{\Pi_1}$ such that $w(\Pi_2) = \Pi_1$, as desired.

Note that for any $\beta \in R \setminus 2R$, there exists a subset Π'' of R with $|\Pi''| = l$ such that $\beta \in \Pi''$ and $\pi(\Pi'')$ is a base of $\pi(R)$. Hence by the above argument, we have (3.2). This completes the proof. $\square$

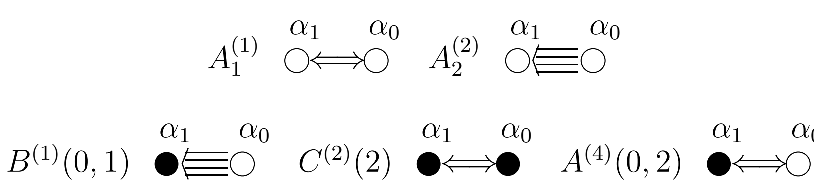
By (3.2), we have

$$
(3.19) \quad \begin{cases} R = W_\Pi \cdot (\Pi \cup (2\Pi \cap R)), \\ (\mathbb{Z}R)^\times \setminus R \\ \quad = W_\Pi \cdot \Big((2\Pi \setminus R) \cup (\bigcup_{r \in 3+\mathbb{Z}_+} r\Pi) \cup ((\mathbb{Z}R)^\times \setminus (\mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi)) \Big). \end{cases}
$$

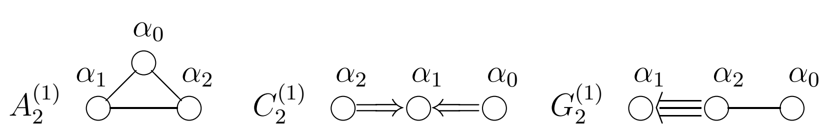
3.2 Dynkin diagrams of affine root systems

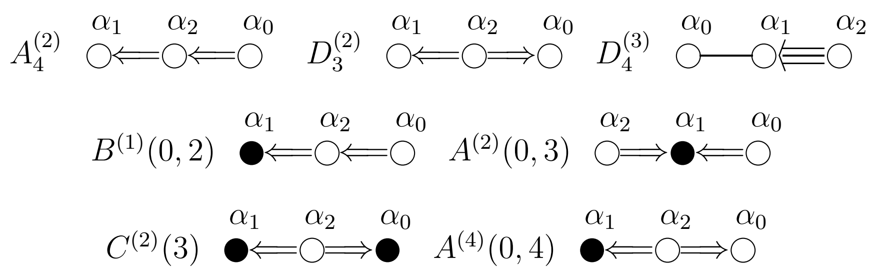
Here we give the Dynkin diagrams for (R, Π) of Theorem 3.1. We assume that if $2\alpha_0 \in R$, then $2\alpha_i \in R$ for some $i \neq 0$, see $A^{(4)}(0, 2l)$ below. We describe them in the same manner as in [11, Table 1-4]; especially, if $2\alpha_i \notin R$ (resp. $2\alpha_i \in R$), then the i-th dot is white (resp. black). The names of them are also the same as in [11, Table 1-4].

(i) The case of $l = 1$:

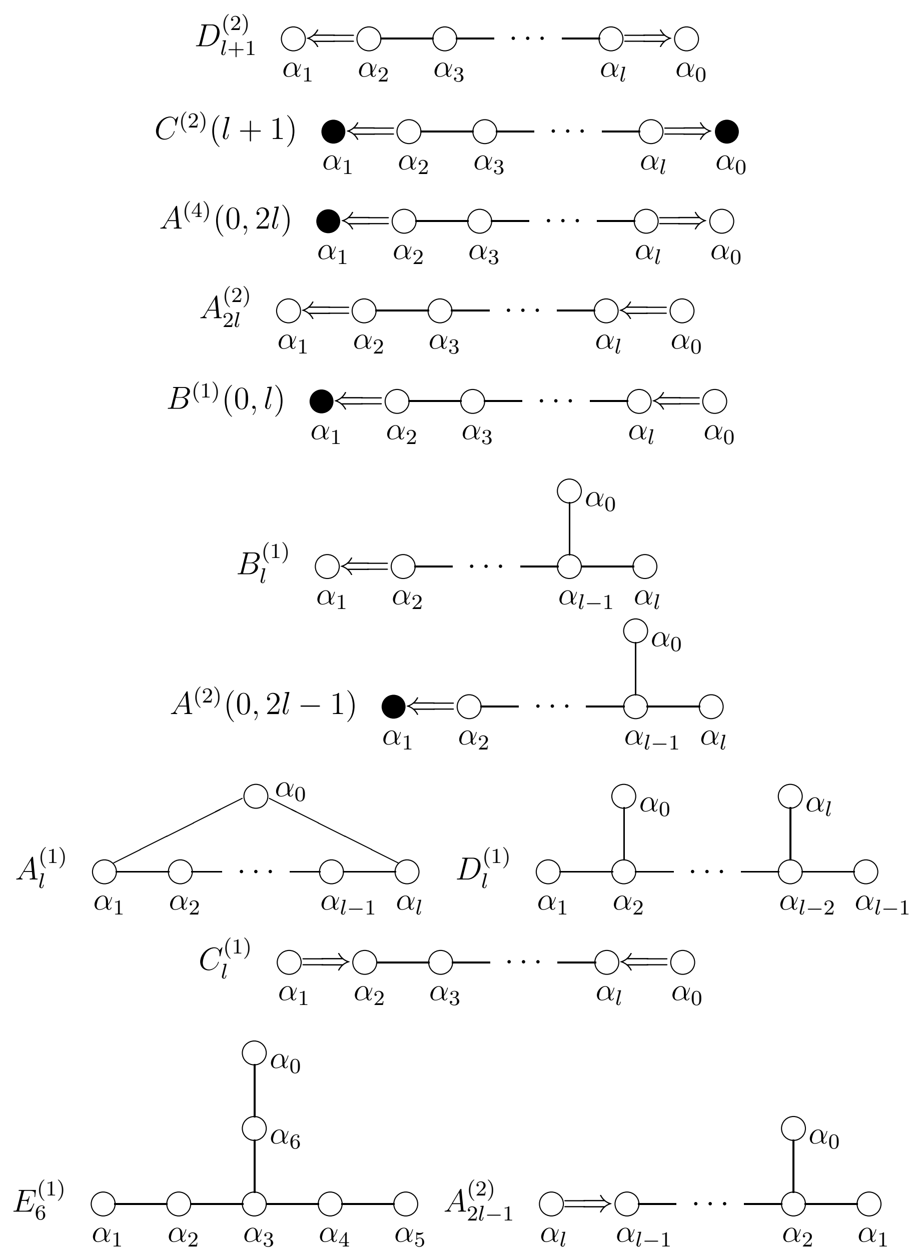


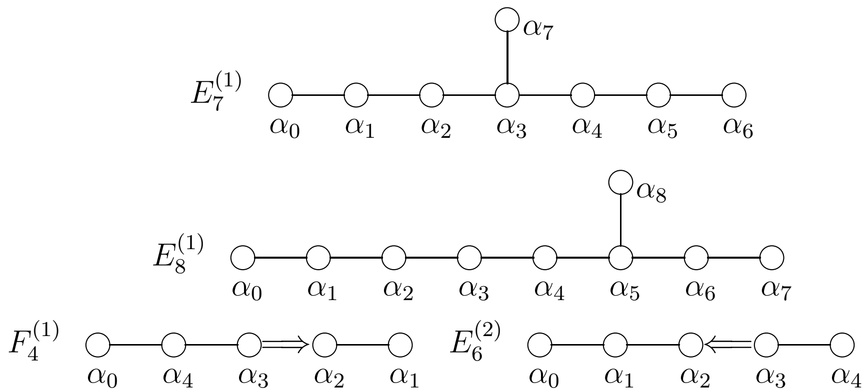
(ii) The case of $l = 2$:





(iii) The case of $l \geq 3$:





4 Elliptic root systems

In this section we assume R is a reduced elliptic root system, that is, $R \cap 2R = \emptyset$ and $n = 2$ (see (2.7)).

4.1 Fundamental-set of an elliptic root system

Definition 4.1. (*Fundamental-set* $\Pi \cup \{a\}$) We say that a subset $\Pi \cup \{a\}$ of $\mathbb{Z}R$ is a *fundamental-set* of R if it satisfies the axioms (FS1)-(FS2) below; we always let

(4.1) $$\pi_a : \mathcal{V} \to \mathcal{V}/\mathbb{R}a$$

denote the canonical map.

(FS1) $a \in (\mathbb{Z}R)^0$ and there exists $b \in (\mathbb{Z}R)^0$ such that $\{a, b\}$ is a basis of $(\mathbb{Z}R)^0$, i.e., $(\mathbb{Z}R)^0 = \mathbb{Z}a \oplus \mathbb{Z}b$.

(FS2) $|\Pi| = l + 1$, $\Pi \subset R$ and $\pi_a(\Pi)$ is a base of the affine root system $\pi_a(R)$.

Until end of this section, let $\Pi \cup \{a\} = \{\alpha_0, \ldots, \alpha_l\} \cup \{a\}$ denote a fundamental-set of R. We assume $\pi(\{\alpha_1, \ldots, \alpha_l\})$ is a base of $\pi(R)$.

Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

(4.2) $$\delta(\Pi) \in \mathbb{N}\Pi \quad \text{and} \quad \mathbb{Z}\delta(\Pi) = (\mathbb{Z}\Pi)^0.$$

Then $\pi_a(\delta(\Pi)) = \delta(\pi_a(\Pi))$ (see (2.10) for $\delta(\pi_a(\Pi))$).

Let $\delta = \delta(\Pi)$ be as in (4.2). By (2.6), (2.11) and (2.8), for $\mathbb{X} \in \{\mathbb{Z}, \mathbb{R}\}$, we have

(4.3) $$\begin{cases} \mathbb{X}R = \bigoplus_{\lambda \in \Pi \cup \{a\}} \mathbb{X}\lambda = (\bigoplus_{\alpha \in \Pi \setminus \{\alpha_0\}} \mathbb{X}\alpha) \bigoplus \mathbb{X}\delta \bigoplus \mathbb{X}a, \\ (\mathbb{X}R)^0 = \mathbb{X}\delta \oplus \mathbb{X}a, \\ R \subset (\mathbb{X}_+\Pi \oplus \mathbb{X}a) \cup (\mathbb{X}_-\Pi \oplus \mathbb{X}a). \end{cases}$$

4.2 Maps k and g

Lemma 4.1. (1) *For any $\alpha \in R$, we have*

$$(\alpha + (\mathbb{Z} \setminus \{0\})a) \cap R \neq \emptyset. \tag{4.4}$$

(2) *Let S be a non-empty proper connected subset of Π. Let $\mathcal{V}^S := \mathbb{R}S \oplus \mathbb{R}a$ and $R^S := R \cap \mathcal{V}^S$. Then $(R^S, \mathcal{V}^S)$ is a reduced affine root system (we have assumed R is reduced), and $(\pi_a(R^S), \mathcal{V}/\mathbb{R}a)$ is an irreducible finite root system with the base $\pi_a(S)$. In particular, $\mathbb{Z}R^S = \mathbb{Z}S \oplus \mathbb{Z}k_S a$ for some $k_S \in \mathbb{N}$.*

Proof. (1) By (4.3), R cannot be included in $\mathbb{Z}\Pi$. Hence there exist $\mu \in R$ and $m \in \mathbb{Z} \setminus \{0\}$ such that $\mu \in ma + \mathbb{Z}\Pi$. Since $\pi_a(R)$ is an affine root system and $\pi_a(\Pi)$ is a base of $\pi_a(R)$, by the first equality of (3.19), there exist $\gamma \in \Pi$, $c \in \{1, 2\}$ and $w \in W_\Pi$ such that $w(\mu) = c\gamma + ma$. Notice that

$$R \ni s_\gamma s_{c\gamma+ma}(\gamma) = s_\gamma(\gamma - (c^{-1}2)(c\gamma + ma)) = \gamma - 2c^{-1}ma. \tag{4.5}$$

(Hence (4.4) holds for this special γ.) Let $\lambda = \gamma - 2c^{-1}ma$. For $\beta \in R$, we have

$$R \ni s_\gamma s_\lambda(\beta) = s_\gamma(\beta - (\gamma^\vee, \beta)\lambda) = \beta + (\gamma^\vee, \beta) \cdot 2c^{-1}ma. \tag{4.6}$$

By (AX5) and (4.3), by repetition of equations similar to (4.6), we see that (4.4) holds for any $\alpha \in R$.

(2) This follows from (1) and (4.3). □

By Lemma 4.1 (2), for each $\alpha \in \Pi$, $R^{\{\alpha\}}$ is a rank-one reduced affine root system and $\{\pi_a(\alpha)\}$ is a base of a rank-one irreducible finite root system $\pi_a(R^{\{\alpha\}})$. By Theorem 3.1, we can define maps

$$k : \Pi \to \mathbb{N} \text{ and } g : \Pi \to \{\emptyset, 2\mathbb{Z} + 1\} \tag{4.7}$$

by

$$R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1,-1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a)) \tag{4.8}$$

($\alpha \in \Pi$) (see also (4.3)).

Since $\pi_a(R) \setminus 2\pi_a(R) = W_{\pi_a(\Pi)} \cdot \pi_a(\Pi)$ (see Theorem 3.1), we have

$$R = \bigcup_{w \in W_\Pi} (\bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a))). \tag{4.9}$$

Since R is determined by Π, k and g,

(4.10) we also denote R by $R(\Pi, k, g)$.

Let $\alpha \in \Pi$. Let $\alpha^* := -\alpha_0(R^{\{\alpha\}}, \{\alpha\}, -k(\alpha)a)$. Then $\alpha^* = c(\alpha)\alpha + k(\alpha)a$, where

$$c(\alpha) = \begin{cases} 1 & \text{if } g(\alpha) = \emptyset, \\ 2 & \text{if } g(\alpha) = 2\mathbb{Z}+1. \end{cases} \tag{4.11}$$

Let $\mathcal{B}_+ := \{\alpha, \alpha^* | \alpha \in \Pi\}$. Then $|\mathcal{B}_+| = 2|\Pi| = 2(l+1)$. By Thereom 3.1, we have

$$R = W_{\mathcal{B}_+} \cdot \mathcal{B}_+ \text{ and } W = W_{\mathcal{B}_+} \tag{4.12}$$

(We have assumed that R is reduced).

Assume $l \geq 2$ (see (2.7)). Let α, $\beta \in \Pi$ be such that $(\beta^\vee, \alpha) = -1$. Let $\gamma = \alpha_0(R^{\{\alpha,\beta\}}, \{\alpha, \beta\}, -k(\alpha)a)$. By Lemma 4.1 (2) and Theorem 3.1, we have $g(\beta) = \emptyset$, $k_{\{\alpha,\beta\}} = k(\alpha)$ and see that $((\beta^\vee, \alpha), k(\beta)/k(\alpha), g(\alpha))$ for the rank-two reduced affine root system $R^{\{\alpha,\beta\}}$ with a base $\{\alpha, \beta, \gamma\}$ is one of the following.

$$\begin{cases} (-1,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } A_2^{(1)}, \text{ and } \gamma = -s_\alpha(\beta^*), \\ (-2,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } B_2^{(1)}, \text{ and } \gamma = -s_\alpha(\beta^*), \\ (-3,1,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } G_2^{(1)}, \text{ and } \gamma = -s_\beta s_\alpha(\beta^*), \\ (-2,2,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } D_3^{(2)}, \text{ and } \gamma = -s_\beta(\alpha^*), \\ (-3,3,\emptyset) & \text{so } R^{\{\alpha,\beta\}} \text{ is } D_4^{(3)}, \text{ and } \gamma = -s_\alpha s_\beta(\alpha^*), \\ (-2,1,2\mathbb{Z}+1) & \text{so } R^{\{\alpha,\beta\}} \text{ is } A_4^{(2)}, \text{ and } \gamma = -s_\beta(\alpha^*). \end{cases} \tag{4.13}$$

4.3 List of (Π, k, g)

Theorem 4.1. *Let $R = R(\Pi, k, g)$ be as in (4.10).*

(1) *Assume $l = 1$. Let $\{\alpha_1, \alpha_0\} = \Pi$ and assume that $\{\pi(\alpha_1)\}$ is a base of $\pi(R)$ and that $k(\alpha_1) \leq k(\alpha_0)$ if $\{\pi(\alpha_0)\}$ is also a base of $\pi(R)$. Then $k(\alpha_1) = 1$ and $((\alpha_0^\vee, \alpha_1), k(\alpha_0), g(\alpha_0), g(\alpha_1))$ is exactly one of the followings:*

$$\begin{array}{l} (-2,1,\emptyset,\emptyset), \\ (-2,1,\emptyset,2\mathbb{Z}+1),\ (-2,1,2\mathbb{Z}+1,\emptyset),\ (-2,1,2\mathbb{Z}+1,2\mathbb{Z}+1), \\ (-2,2,\emptyset,\emptyset),\ (-2,2,2\mathbb{Z}+1,\emptyset), \\ (-1,1,\emptyset,\emptyset),\ (-1,1,\emptyset,2\mathbb{Z}+1),\ (-1,2,\emptyset,\emptyset),\ (-1,2,\emptyset,2\mathbb{Z}+1), \\ (-1,4,\emptyset,\emptyset). \end{array} \tag{4.14}$$

(2) *Assume $l \geq 2$. Then there exists $R(\Pi, k, g)$ such that $(W_\Pi \cdot \Pi, \mathbb{R}\Pi)$ is a rank-l reduced affine root system of any type with a base Π and $k : \Pi \to \mathbb{N}$ and $g : \Pi \to \{\emptyset, 2\mathbb{Z}+1\}$ are any maps satisfying the condition that $1 \in k(\Pi)$ and $((\alpha^\vee, \beta), k(\beta)/k(\alpha), g(\alpha))$ is the same as one of (4.13) for any α, $\beta \in \Pi$ with $(\beta^\vee, \alpha) = -1$.*

The statements of this theorem is well-known and, however, some of $R(\Pi, k, g)$'s are isomorphic (see [16, (6.6)] and [1, Lists 4.6, 4.25, 4.67, 4.78]). For the case $l \geq 2$, which of them are isomorphic can be read off from the statement of Theorem 6.1.

5 Elliptic Lie algebras with rank ≥ 2

In this section we assume R is a reduced elliptic root system with rank ≥ 2, that is, $R \cap 2R = \emptyset$, $n = 2$ and $l \geq 2$ (see (2.7)). We have assumed the rank $l \geq 2$ mainly because we use the fact (5.6) below. We fix a fundamental-set $\Pi \cup \{a\}$ of R.

5.1 Useful lemma

The following lemma is useful.

Lemma 5.1. *Let $\mathcal{V}'$ be a 2-dimensional $\mathbb{C}$-linear space having a non-degenerate symmetric bilinear form $(\ ,\ ) : \mathcal{V}' \times \mathcal{V}' \to \mathbb{C}$. Let γ_1, $\gamma_2 \in (\mathcal{V}')^\times$. Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$ generated by $\bar{h}_\gamma$ ($\gamma \in \mathcal{V}'$), $\bar{E}_1$, $\bar{E}_2$, $\bar{F}_1$, $\bar{F}_2$ and satisfying the equations $\bar{h}_{x\gamma+x'\gamma'} = x\bar{h}_\gamma + x'\bar{h}_{\gamma'}$, $[\bar{h}_\gamma, \bar{h}_{\gamma'}] = 0$, $[\bar{h}_\gamma, \bar{E}_i] = (\gamma, \gamma_i)\bar{E}_i$, $[\bar{h}_\gamma, \bar{F}_i] = -(\gamma, \gamma_i)\bar{F}_i$, and $[\bar{E}_i, \bar{F}_i] = \delta_{ij}\bar{h}_{\gamma_i^\vee}$, for x, $x' \in \mathbb{C}$, γ, $\gamma' \in \mathcal{V}'$, and $i \in \{1, 2\}$.*

(1) *For $k \in \mathbb{N}$, we have*

$$\begin{aligned} &[\mathrm{ad}(\bar{E}_1)^k(\bar{E}_2), \mathrm{ad}(\bar{F}_1)^k(\bar{F}_2)] \\ &\quad = k!(\textstyle\prod_{m=1}^{k-1}((\gamma_1^\vee, \gamma_2) + m))(k(\gamma_1, \gamma_2^\vee)\bar{h}_{\gamma_1^\vee} + (\gamma_1^\vee, \gamma_2)\bar{h}_{\gamma_2^\vee}). \end{aligned} \tag{5.1}$$

(2) *Let $m := (\gamma_1^\vee, \gamma_2)$. Assume $m \in \mathbb{Z}_-$. Assume that $\bar{h}_{\gamma_1^\vee}$ and $\bar{h}_{\gamma_2^\vee}$ are linearly independent. Assume $\mathrm{ad}(\bar{E}_1)^r(\bar{E}_2) = \mathrm{ad}(\bar{F}_1)^r(\bar{F}_2) = 0$ for some $r \in \mathbb{N}$. Let*

$$\bar{n} = n(\bar{E}_1, \bar{F}_1) := \exp(\mathrm{ad}\bar{E}_1)\exp(-\mathrm{ad}\bar{F}_1)\exp(\mathrm{ad}\bar{E}_1). \tag{5.2}$$

Then we have

$$\begin{aligned} &\mathrm{ad}(\bar{E}_1)^{1-m}(\bar{E}_2) = \mathrm{ad}(\bar{F}_1)^{1-m}(\bar{F}_2) = 0, \\ &\bar{n}(\bar{h}_\gamma) = \bar{h}_\gamma - (\gamma_1, \gamma)\bar{h}_{\gamma_1^\vee},\ \bar{n}(\bar{E}_1) = -\bar{E}_1,\ \bar{n}(\bar{F}_1) = -\bar{F}_1, \\ &\bar{n}((\mathrm{ad}\bar{E}_1)^i\bar{E}_2) = \tfrac{(-1)^i i!}{(-m-i)!}(\mathrm{ad}\bar{E}_1)^{-m-i}\bar{E}_2 \neq 0, \\ &\bar{n}((\mathrm{ad}\bar{F}_1)^i\bar{F}_2) = \tfrac{(-1)^{m-i} i!}{(-m-i)!}(\mathrm{ad}\bar{F}_1)^{-m-i}\bar{F}_2 \neq 0, \end{aligned} \tag{5.3}$$

for $0 \leq i \leq -m$ and $\gamma \in \mathcal{V}'$.

We can get (5.1) directly and get (5.3) by using a representation theory of sl_2.

5.2 Definition of elliptic Lie algebras with rank ≥ 2

Let $\mathcal{A} := \{(\alpha, \beta) \in \Pi \times \Pi \,|\, (\alpha, \beta^\vee) = -1\}$. Let $\mathcal{B} := \mathcal{B}_+ \cup (-\mathcal{B}_+)$, and $\mathcal{B}^{2,\prime} := \{(\mu, \nu) \in \mathcal{B} \times \mathcal{B} \,|\, \mu \neq \nu \neq -\mu\}$. For $(\mu, \nu) \in \mathcal{B}^{2,\prime}$, let $x_{\mu,\nu} = 1 - ((\mu^\vee, \nu) - |(\mu^\vee, \nu)|)/2$. Let $\mathcal{V}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{V}$, so $\mathcal{V}^{\mathbb{C}}$ is a $l + 2$-dimensional $\mathbb{C}$-linear space. We identify $\mathcal{V}$ with the $\mathbb{R}$-linear subspace $1 \otimes \mathcal{V}$ of $\mathcal{V}^{\mathbb{C}}$. We say that a map $\omega : \mathcal{A} \to \mathbb{C}^\times$ is a *tuning* if $\omega(\alpha, \beta)\omega(\beta, \alpha) = 1$ whenever $(\alpha^\vee, \beta) = -1$. Denote ω_1 by the tuning with $\omega_1(\alpha, \beta) = 1$ for all $(\alpha, \beta) \in \mathcal{A}$, and moreover, if $W_\Pi \cdot \Pi$ is

$A_l^{(1)}$, then for $q \in \mathbb{C}^\times$, denote ω_q by the tuning with $\omega(\alpha_i, \alpha_{i+1}) = 1$ $(0 \le i \le l)$ and $\omega(\alpha_l, \alpha_0) = q$, where the numbering of the elements of Π is the same as that of the Dynkin diagram of $A_l^{(1)}$ in Subsection 3.2.

Definition 5.1. Let $\mathfrak{g}^\omega = \mathfrak{g}(\Pi, k, g, \omega)$ be the Lie algebra over $\mathbb{C}$ defined by generators:

$$h_\sigma \ (\sigma \in \mathcal{V}^{\mathbb{C}}), \quad E_\mu \ (\mu \in \mathcal{B}),$$

and relations:

(SR1) $xh_\sigma + yh_\tau = h_{x\sigma+y\tau}$ if $x, y \in \mathbb{C}$ and $\sigma, \tau \in \mathcal{V}^{\mathbb{C}}$,
(SR2) $[h_\sigma, h_\tau] = 0$ if $\sigma, \tau \in \mathcal{V}^{\mathbb{C}}$,
(SR3) $[h_\sigma, E_\mu] = (\sigma, \mu)E_\mu$ if $\sigma \in \mathcal{V}^{\mathbb{C}}$ and $\mu \in \mathcal{B}$,
(SR4) $[E_\mu, E_{-\mu}] = h_{\mu^\vee}$ if $\mu \in \mathcal{B}_+$,
(SR5) $(\mathrm{ad}E_\mu)^{x_{\mu,\nu}} E_\nu = 0$ if $(\mu, \nu) \in \mathcal{B}^{2,\prime}$,
(SR6) $c(\alpha)(\mathrm{ad}E_{\alpha^*})^{\frac{k(\beta)}{k(\alpha)}} E_\beta = \omega(\alpha, \beta)(\mathrm{ad}E_\alpha)^{c(\alpha)\frac{k(\beta)}{k(\alpha)}} E_{\beta^*}$ if $(\alpha, \beta) \in \mathcal{A}$,
(SR7) $(-1)^{c(\alpha)+1}c(\alpha)(\mathrm{ad}E_{-\alpha^*})^{\frac{k(\beta)}{k(\alpha)}} E_{-\beta} = \frac{1}{\omega(\alpha,\beta)}(\mathrm{ad}E_{-\alpha})^{c(\alpha)\frac{k(\beta)}{k(\alpha)}} E_{-\beta^*}$ if $(\alpha, \beta) \in \mathcal{A}$,
(SR8) $(\mathrm{ad}E_\alpha)^i(\mathrm{ad}E_{\alpha^*})^{\frac{k(\beta)}{k(\alpha)}-i} E_\beta = 0$ if $(\alpha, \beta) \in \mathcal{A}$ and $1 \le i \le \frac{k(\beta)}{k(\alpha)} - 1$,
(SR9) $(\mathrm{ad}E_{-\alpha})^i(\mathrm{ad}E_{-\alpha^*})^{\frac{k(\beta)}{k(\alpha)}-i} E_{-\beta} = 0$ if $(\alpha, \beta) \in \mathcal{A}$ and $1 \le i \le \frac{k(\beta)}{k(\alpha)} - 1$.

We call $\mathfrak{g}(\Pi, k, g, \omega)$ an *elliptic Lie algebra*, see Introduction. Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g) := \mathfrak{g}^{\omega_1}$.

We have

Lemma 5.2. *If $W_\Pi \cdot \Pi$ is not $A_l^{(1)}$ (resp. is $A_l^{(1)}$), then there is an isomorphism φ from $\mathfrak{g}^\omega$ to $\mathfrak{g}$ (resp. to $\mathfrak{g}^{\omega_q}$ for some $q \in \mathbb{C}^\times$) such that $\varphi(h_\sigma) = h_\sigma$ $(\sigma \in \mathcal{V}^{\mathbb{C}})$ and $\varphi(E_\mu) \in \mathbb{C}^\times E_\mu$ $(\mu \in \mathcal{B})$.*

Proof. Using (5.1), we can modify (SR6-7) by taking non-zero scalar products of E_μ's. □

Let $\mathfrak{h}^\omega = \mathfrak{h}^\omega(\Pi, k, g, \omega) := \{h_\sigma \in \mathfrak{g}^\omega | \sigma \in \mathcal{V}^{\mathbb{C}}\}$, and $\mathfrak{h} = \mathfrak{h}(\Pi, k, g) := \mathfrak{h}^{\omega_1}$.

Since all equations in (SR1-9) are $\mathbb{Z}R$-homogeneous, where $R = R(\Pi, k, g)$, we can regard $\mathfrak{g}^\omega$ as the $\mathbb{Z}R$-graded Lie algebra $\mathfrak{g}^\omega = \oplus_{\sigma\in\mathbb{Z}R}\mathfrak{g}^\omega_\sigma$ (that is $[\mathfrak{g}^\omega_\sigma, \mathfrak{g}^\omega_{\sigma'}] \subset \mathfrak{g}^\omega_{\sigma+\sigma'}$) such that $E_\mu \in \mathfrak{g}^\omega_\mu$ for all $\mu \in \mathcal{B}$. Note $\mathfrak{h}^\omega \subset \mathfrak{g}^\omega_0$. For each $\mu \in \mathcal{B}_+$, we can define n_μ to be $n(E_\mu, E_{-\mu})$ (see (5.2)) as an automorphism of $\mathfrak{g}^\omega$, so $n_\mu(\mathfrak{g}^\omega_\sigma) = \mathfrak{g}^\omega_{s_\mu(\sigma)}$. Let $\mathcal{R}^\omega = \{\sigma \in \mathbb{Z}R | \dim \mathfrak{g}^\omega_\sigma \ne 0\}$. Then we have

$$W_{\mathcal{B}_+} \cdot \mathcal{R}^\omega = \mathcal{R}^\omega. \tag{5.4}$$

Let S a non-empty proper connected subset of Π. Let $\mathfrak{g}^{\omega,S}$ be the Lie algebra over $\mathbb{C}$ defined by the generators h_σ $(\sigma \in \mathbb{C}S \oplus \mathbb{C}a)$, $E_{\pm\alpha}$, $E_{\pm\alpha^*}$ $(\alpha \in S)$ and the same

relations as those in (SR1-9). Let $\iota^{\omega,S} : \mathfrak{g}^{\omega,S} \to \mathfrak{g}^{\omega}$ be the homomorphism sending the generators to those denoted by the same symbols. Let $\mathfrak{g}^{\omega,S}_{\sigma} = (\iota^{\omega,S})^{-1}(\mathfrak{g}^{\omega}_{\sigma})$ for $\sigma \in \mathbb{Z}R^S$, so $\mathfrak{g}^{\omega,S} = \oplus_{\sigma\in\mathbb{Z}R^S}\mathfrak{g}^{\omega,S}_{\sigma}$. Let $\mathfrak{g}^S = \mathfrak{g}^{\omega_1,S}$, and $\mathfrak{g}^S_{\sigma} = \mathfrak{g}^{\omega_1,S}_{\sigma}$. Let $\mathcal{R}^{\omega,S} = \{\sigma \in \mathbb{Z}R^S | \dim \mathfrak{g}^{\omega,S}_{\sigma} \neq 0\}$.

Let $\alpha \in \Pi$. Then $\mathfrak{g}^{\omega,\{\alpha\}} = \mathfrak{g}^{\{\alpha\}}$, since $\mathfrak{g}^{\omega,\{\alpha\}}$ is defined by using (SR1-5). By Serre's relations (SR1-5), $\mathfrak{g}^{\omega,\{\alpha\}}$ is (the derived algebra of) an affine Lie algebra with $\mathcal{R}^{\omega,\{\alpha\}} = R^{\{\alpha\}} \cup \mathbb{Z}k(\alpha)a$, where the affine root system $R^{\{\alpha\}}$ is $A^{(1)}_1$ or $A^{(1)}_2$. Hence $\dim \mathfrak{g}^{\omega,\{\alpha\}}_0 = 2$, and $\dim \mathfrak{g}^{\omega,\{\alpha\}}_{\lambda} = 1$ ($\lambda \in \mathcal{R}^{\omega,\{\alpha\}} \setminus \{0\}$). Note $\mathcal{R}^{\omega,\{\alpha\}} \setminus \{0\} = R^{\{\alpha\}} \cup \mathbb{Z}^{\times}k(\alpha)a$.

Lemma 5.3. *There is a homomorphism χ^{ω} from $\mathfrak{g}^{\omega}$ to a Lie algebra $\mathfrak{b}^{\omega}$ such that* $\dim \chi^{\omega}(\mathfrak{h}^{\omega}) = l + 2$, $\dim \chi^{\omega}(\iota^{\omega,\{\alpha\}}(\mathfrak{g}^{\omega,\{\alpha\}}_{\lambda})) = 1$ *for all $\alpha \in \Pi$ and all $\lambda \in R^{\{\alpha\}} \cup \mathbb{Z}^{\times}k(\alpha)a$, and*

$$\begin{aligned}\chi^{\omega}(\mathfrak{h}^{\omega} &+ \textstyle\sum_{\alpha\in\Pi}\sum_{\lambda\in R^{\{\alpha\}}\cup\mathbb{Z}^{\times}k(\alpha)a}\iota^{\omega,\{\alpha\}}(\mathfrak{g}^{\omega,\{\alpha\}}_{\lambda})) \\ &= \chi^{\omega}(\mathfrak{h}^{\omega}) \oplus \textstyle\bigoplus_{\alpha\in\Pi}\bigoplus_{\lambda\in R^{\{\alpha\}}\cup\mathbb{Z}^{\times}k(\alpha)a}\chi^{\omega}(\iota^{\omega,\{\alpha\}}(\mathfrak{g}^{\omega,\{\alpha\}}_{\lambda})).\end{aligned} \tag{5.5}$$

(If $\omega = \omega_1$, then $\mathfrak{b}^{\omega}$ is given as an 'affinization' $\mathfrak{a}\otimes\mathbb{C}[t, t^{-1}]\oplus\mathbb{C}c$ of (the derived algebra of) an affine Lie algebra $\mathfrak{a}$, see [19, Proposition 3.1].)

Proof. If $\omega = \omega_1$, then we can define $\chi = \chi^{\omega_1}$ in a way entirely similar to that of [19, Proposition 3.1], inspired by so-called an 'unfolding process' of a Dynkin diagram of a reduced affine root system, and we see by checking each case directly that such χ has the property (5.5). The existence of a χ^{ω_q} is well-known (see [6]). Then this lemma follows from Lemma 5.2. □

For each $\alpha \in \Pi$, let $[R^{\{\alpha\}}]^+ := R^{\{\alpha\}}\cap(\mathbb{N}\alpha+\mathbb{Z}k(\alpha)a)$, and $[R^{\{\alpha\}}]^- := -[R^{\{\alpha\}}]^+$. Note that $R^{\{\alpha\}} = [R^{\{\alpha\}}]^+ \cup [R^{\{\alpha\}}]^-$.

Lemma 5.4. *For each $(\alpha, \beta) \in \mathcal{A}$,*

$$\begin{aligned}&\mathfrak{g}^{\omega,\{\alpha,\beta\}} \textit{ is (the derived algebra of) an affine Lie algebra} \\ &\textit{with the affine root system } R^{\{\alpha,\beta\}},\end{aligned} \tag{5.6}$$

which implies $\mathcal{R}^{\omega,\{\alpha,\beta\}} = R^{\{\alpha,\beta\}} \cup \mathbb{Z}k(\alpha)a$. In particular, for each $(\alpha', \beta') \in \Pi\times\Pi$ with $\alpha' \neq \beta'$, we have

$$[\iota^{\omega,\{\alpha'\}}(\mathfrak{g}^{\omega,\{\alpha'\}}_{\lambda}), \iota^{\omega,\{\beta'\}}(\mathfrak{g}^{\omega,\{\beta'\}}_{\mu})] = 0 \tag{5.7}$$

for all $(\lambda, \mu) \in ([R^{\{\alpha'\}}]^+ \times [R^{\{\beta'\}}]^-) \cup ([R^{\{\alpha'\}}]^- \times [R^{\{\beta'\}}]^+)$.

Proof. Note first that h_α, h_β and h_a are linearly independent in $\mathfrak{g}^{\omega,\{\alpha,\beta\}}$, which follows from Lemma 5.3. Let $\gamma \in R^{\{\alpha,\beta\}}$ be as in (4.13). If γ is expressed as $-s_{\gamma_1}\ldots s_{\gamma_{r-1}}(\gamma^*_r)$ in (4.13) with $\gamma_i \in \{\alpha, \beta\}$, then we let $E_{\pm\gamma} := n_{\gamma_1}\ldots n_{\gamma_{r-1}}(E_{\mp\gamma^*_r}) \in \mathfrak{g}^{\omega,\{\alpha,\beta\}}_{\pm\gamma}$. Let $\gamma_{r+1} \in \{\alpha, \beta\} \setminus \{\gamma_r\}$. By (SR6-7) and (5.3), we

have $n_{\pm\gamma_r^*}(E_{\pm\gamma_{r+1}}) = n_{\pm\gamma_r}(E_{\pm\gamma_{r+1}^*})$. Hence $\mathfrak{g}^{\omega,\{\alpha,\beta\}}$ is generated by $E_{\pm\alpha}$, $E_{\pm\beta}$ and $E_{\pm\gamma}$. We show

$$[E_{\pm\alpha}, E_{\mp\gamma}] = [E_{\pm\beta}, E_{\mp\gamma}] = 0. \tag{5.8}$$

If $R^{\{\alpha,\beta\}} \neq A_4^{(2)}$, we have this in the same way as in [19, §2.3]. Assume $R^{\{\alpha,\beta\}} = A_4^{(2)}$. We write $X \sim Y$ if $X \in \mathbb{C}^\times Y$. By (5.3) and (SR6),

$$E_{-\gamma} \sim [E_\beta, [E_\beta, E_{\alpha^*}]] \sim [E_\beta, [E_\alpha, [E_\alpha, E_{\beta^*}]]] \tag{5.9}$$

Then $[E_\beta, E_{-\gamma}] = 0$ follows from (SR5). We have

$$\begin{aligned}
[E_{-\gamma}, E_\alpha] &\sim [[E_\beta, [E_\alpha, [E_\alpha, E_{\beta^*}]]], E_\alpha] \quad \text{(by (5.9))} \\
&\sim [[E_\beta, E_\alpha], [E_\alpha, [E_\alpha, E_{\beta^*}]]]] \quad \text{(by (SR5))} \\
&\sim [[E_\beta, E_\alpha], [E_\beta, E_{\alpha^*}]]] \quad \text{(by (SR6))} \\
&\sim n_\beta([E_\alpha, [E_\beta, E_{\alpha^*}]]) \quad \text{(by (5.3))} \\
&\sim n_\beta([E_\alpha, [E_\alpha, [E_\alpha, E_{\beta^*}]]]) \quad \text{(by (SR6))} \\
&= 0 \quad \text{(by (SR5))}.
\end{aligned}$$

The remaining equalities of (5.8) can be shown similarly. Hence by (5.3) and (SR5), the above generators satisfy Serre's relations. Hence (5.6) holds, as desired. □

For $i \in \mathbb{N}$, let $(\mathfrak{n}^{\omega,\pm})^{(i)}$ be the $\mathbb{C}$-linear subspaces of $\mathfrak{g}^\omega$ defined by $(\mathfrak{n}^{\omega,\pm})^{(1)} := \oplus_{\alpha\in\Pi}\oplus_{\lambda\in[R^{\{\alpha\}}]^\pm}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\lambda^{\omega,\{\alpha\}})$ (see Lemme 5.3), and $(\mathfrak{n}^{\omega,\pm})^{(i)} := [(\mathfrak{n}^{\omega,\pm})^{(1)}, (\mathfrak{n}^{\omega,\pm})^{(i-1)}]$ inductively for $i \geq 2$. Let $\mathfrak{n}^{\omega,\pm}$ be the two Lie subalgebras of $\mathfrak{g}^\omega$ defined by $\mathfrak{n}^{\omega,\pm} := \sum_{i=1}^\infty(\mathfrak{n}^{\omega,\pm})^{(i)}$. Let $\mathfrak{n}_\sigma^{\omega,\pm} = \mathfrak{g}_\sigma^\omega \cap \mathfrak{n}^{\omega,\pm}$. Then $\mathfrak{n}^{\omega,\pm} = \oplus_{\sigma\in(\mathbb{Z}_\pm\Pi\oplus\mathbb{Z}a)\setminus\mathbb{Z}a}\mathfrak{n}_\sigma^{\omega,\pm}$. For each $\alpha \in \Pi$, since $\iota^{\omega,\{\alpha\}}$ is a Lie algebra homomorphism (preserving $\mathbb{Z}\Pi\oplus\mathbb{Z}a$-grading), we have $\mathfrak{n}_\mu^{\omega,\pm} = \mathfrak{n}_\mu^{\omega,\pm} \cap (\mathfrak{n}^{\omega,\pm})^{(1)} = \iota^{\omega,\{\alpha\}}(\mathfrak{g}_\mu^{\omega,\{\alpha\}})$ for all $\mu \in (\mathbb{Z}_\pm\alpha \oplus \mathbb{Z}a) \setminus \mathbb{Z}a$. Moreover, by (5.7), we have

$$[(\mathfrak{n}^{\omega,+})^{(1)}, (\mathfrak{n}^{\omega,-})^{(1)}] \subset (\mathfrak{n}^{\omega,+})^{(1)} + (\mathfrak{n}^{\omega,-})^{(1)} + \sum_{\alpha\in\Pi}\sum_{\sigma\in\mathbb{Z}k(\alpha)a}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}}). \tag{5.10}$$

Hence by Lemma 5.3 and (5.6), we have

$$\mathfrak{g}^\omega = \mathfrak{h}^\omega \oplus \mathfrak{n}^{\omega,+} \oplus \mathfrak{n}^{\omega,-} \oplus (\bigoplus_{\alpha\in\Pi}\bigoplus_{\sigma\in\mathbb{Z}^\times k(\alpha)a}\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}})), \tag{5.11}$$

$\dim\mathfrak{h}^\omega = l+2$, and $\dim\mathfrak{n}_\lambda^{\omega,\pm} = \dim\iota^{\omega,\{\alpha\}}(\mathfrak{g}_\sigma^{\omega,\{\alpha\}}) = 1$ for $\alpha \in \Pi$, $\lambda \in [R^{\{\alpha\}}]^\pm$ and $\sigma \in \mathbb{Z}^\times k(\alpha)a$. By (3.19), we have

(5.12)
$$\begin{cases} R = W_\Pi \cdot \bigcup_{\alpha\in\Pi}[R^{\{\alpha\}}]^+, \\ (\mathbb{Z}R)^\times \setminus R \\ = W_\Pi \cdot (\bigcup_{\alpha\in\Pi}(\mathbb{N}\alpha\oplus\mathbb{Z}a) \setminus [R^{\{\alpha\}}]^+) \cup ((\mathbb{Z}R)^\times \setminus (\mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi) \oplus \mathbb{Z}a). \end{cases}$$

Then by (5.4), using a standard argument as in [10], [18], together with the automorphisms n_μ ($\mu \in \mathcal{B}_+$), we have

Theorem 5.1. *We have* $(\mathcal{R}^\omega)^\times = R$, $\dim \mathfrak{g}^\omega_\mu = 1$ $(\mu \in R)$, $\mathfrak{g}^\omega_0 = \mathfrak{h}^\omega$, $\dim \mathfrak{h}^\omega = l+2$, $(\mathcal{R}^\omega)^0 \subset \mathbb{Z}\delta \oplus \mathbb{Z}a$, *and* $\dim \mathfrak{g}^\omega_{ma} = |\{\alpha \in \Pi | m \in \mathbb{Z}k(\alpha)\}|$ $(m \in \mathbb{Z}^\times)$.

By the following theorem, we can compute $\dim \mathfrak{g}^\omega_\lambda$ for $\lambda \in \mathbb{Z}\delta \oplus \mathbb{Z}a$.

Theorem 5.2. *Let* $\Pi' \cup \{a'\}$ *be a fundamental-set of* R*. Then there exist a tuning* η *for* $\Pi' \cup \{a'\}$ *and an isomorphism* $f : \mathfrak{g}(\Pi', k', g', \eta) \to \mathfrak{g}^\omega$ *such that* $f(\mathfrak{g}'^{;\eta}_\lambda) = \mathfrak{g}^\omega_\lambda$ *for all* $\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}a$*, where* $\mathfrak{g}'^{,\eta} := \mathfrak{g}(\Pi', k', g', \eta)$*. In particular, we have*

$$\dim \mathfrak{g}^\omega_{ma'} = |\{\alpha' \in \Pi' | m \in \mathbb{Z}k'(\alpha')\}| \text{ for } m \in \mathbb{Z}^\times. \tag{5.13}$$

Proof. Let ${\mathcal{B}_+}' = \{\alpha', (\alpha')^* | \alpha \in \Pi'\}$ and $\mathcal{B}' = {\mathcal{B}_+}' \cup -{\mathcal{B}_+}'$. By (SR1-9), Theorem 5.1 and (5.3), for some η, we have a homomorphism f of the statement such that $f(\mathfrak{g}'^{;\eta}_{\mu'}) = \mathfrak{g}^\omega_{\mu'}$ for all $\mu' \in \mathcal{B}'$. Since $\mathfrak{g}'^{,\eta}$ is generated by $\mathfrak{g}'^{;\eta}_{\mu'}$ $(\mu' \in \mathcal{B}')$, we have $f(\mathfrak{g}'^{;\eta}_\lambda) \subset \mathfrak{g}^\omega_\lambda$ for all $\lambda \in \mathbb{Z}R = \mathbb{Z}\Pi' \oplus \mathbb{Z}a'$. Since $R = W_{{\mathcal{B}_+}'} \cdot {\mathcal{B}_+}'$ by (4.12), using $n(E_{\mu'}, E_{-\mu'}) \in \mathrm{Aut}(\mathfrak{g}'^{,\eta})$ $(\mu' \in \mathcal{B}')$, by Theorem 5.1, we have $f(\mathfrak{g}'^{;\eta}_\beta) = \mathfrak{g}^\omega_\beta$ for all $\beta \in R$. Since $E_\mu \in f(\mathfrak{g}'^{,\eta})$ for all $\mu \in \mathcal{B}$, we have $f(\mathfrak{g}'^{,\eta}) = \mathfrak{g}^\omega$, so $f(\mathfrak{g}'^{;\eta}_\lambda) = \mathfrak{g}^\omega_\lambda$ for all $\lambda \in \mathbb{Z}R$. By the same argument, for some tuning ω' for $\Pi \cup \{a\}$, we have an epimorphism $f' : \mathfrak{g}^{\omega'} = \mathfrak{g}(\Pi, k, g, \omega') \to \mathfrak{g}'^{,\eta}$ such that $f'(\mathfrak{g}^{\omega'}_\lambda) = \mathfrak{g}'^{;\eta}_\lambda$ for all $\lambda \in \mathbb{Z}R$. Hence $\dim \mathfrak{g}^{\omega'}_\lambda \geq \dim \mathfrak{g}^\omega_\lambda$ for all $\mathbb{Z}R$, so $(\mathcal{R}^\omega)^0 \subset (\mathcal{R}^{\omega'})^0$. Assume that $W_\Pi \cdot \Pi$ is not $A^{(1)}_l$. By Lemma 5.2, we have $\dim \mathfrak{g}^{\omega'}_\lambda = \dim \mathfrak{g}_\lambda = \dim \mathfrak{g}^\omega_\lambda$ for all $\lambda \in \mathbb{Z}R$, so $(\mathcal{R}^\omega)^0 = (\mathcal{R}^{\omega'})^0$. Hence $f \circ f'$ is an isomorphism, so is f. Assume that $W_\Pi \cdot \Pi$ is $A^{(1)}_l$. Assume $\varphi : \mathfrak{g}(\Pi, k, g, \omega_{q_1}) \to \mathfrak{g}(\Pi, k, g, \omega_{q_2})$ is an epimorphism such that $\varphi(\mathfrak{g}(\Pi, k, g, \omega_{q_1})_\lambda) = \mathfrak{g}(\Pi, k, g, \omega_{q_2})_\lambda$ for all $\lambda \in \mathbb{Z}R$. For $\gamma \in \mathcal{B}_+$, let $c_\gamma \in \mathbb{C}^\times$ be such that $\varphi(E_\gamma) = c_\gamma E_\gamma$ ($E_\gamma \neq 0$ by Lemma 5.3). For $\alpha \in \Pi$, let $d_\alpha = c_\alpha / c_{\alpha^*}$. By (SR6), we have $\omega_{q_2}(\alpha, \beta) = \omega_{q_1}(\alpha, \beta) d_\alpha / d_\beta$ (the element of (SR6) is not zero by Lemma 5.3 and (5.1)). Hence $d_{\alpha_i} = d_{\alpha_{i+1}}$ for $0 \leq i \leq l$. Since $\omega_{q_2}(\alpha_l, \alpha_0) = \omega_{q_1}(\alpha_l, \alpha_0)$, we have $q_1 = q_2$. Then by the same argument as above, we conclude that f is an isomorphism.

The last statement follows from Theorem 5.1. □

6 List of $\dim \mathfrak{g}_{m\delta + ra}$

In this section we use the notation as follows. For a $\mathbb{Z}$-module X, $r \in \mathbb{Z}$ and $x, y \in X$, let $x \equiv_r y$ means $x - y \in rX$. Recall that $l = |\Pi| - 1 \geq 2$, and see Subsection 3.2 for the numbering of the elements α_i $(0 \leq i \leq l)$ of Π. Let $\delta = \delta(\Pi)$. Fix $\gamma_1 \in \Pi_{\mathrm{sh}} \setminus \{\alpha_0\}$. Fix $\gamma_2 \in \Pi_{\mathrm{lg}} \setminus \{\alpha_0\}$ if $R_{\mathrm{lg}} \neq \emptyset$. Let $M := \mathbb{Z}\delta \oplus \mathbb{Z}a$. We also denote $m\delta + ra \in M$ with $m, r \in \mathbb{Z}$ by $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]$. Let $R = R(\Pi, k, g)$ be as in (4.10). Let L_{sh}, L_{lg} and L_{ex} be the subsets of M such that $\gamma_1 + L_{\mathrm{sh}} = R \cap (\gamma_1 + M)$, $\gamma_2 + L_{\mathrm{lg}} = R \cap (\gamma_2 + M)$ (if $R_{\mathrm{lg}} \neq \emptyset$), and $2\gamma_1 + L_{\mathrm{ex}} = R \cap (2\gamma_1 + M)$ (if

$R_{\rm ex} \neq \emptyset$). Let $\Pi' := \Pi \setminus \{\alpha_0\}$, so $\pi(\Pi')$ is a base of $\pi(R)$. By Lemma 2.1, we have $R_{\rm sh} = W_{\Pi'} \cdot \gamma_1 + L_{\rm sh}$, $R_{\rm lg} = W_{\Pi'} \cdot \gamma_2 + L_{\rm lg}$ and $R_{\rm ex} = W_{\Pi'} \cdot 2\gamma_1 + L_{\rm ex}$. Let $\mathfrak{g}^\omega := \mathfrak{g}(\Pi, k, g, \omega)$, and $\mathfrak{g} := \mathfrak{g}^{\omega_1}$.

Remark 6.1. (Due to Kaiming Zhao) Here we would like to mention that a map from M to $\{0, 1, \dots, t-1\}$ which is periodic modulo t on any line in M is not necessarily meant to be periodic modulo tM. This indicates that we have to be very careful when calculating $\dim \mathfrak{g}^\omega_{m\delta+ra}$ because (5.13) does not immediately imply that $\dim \mathfrak{g}^\omega_{m\delta+ra}$ is periodic, although we finally see that this is true.

Let $f : M \to \mathbb{Z}_+$ be a map such that $m\mathbb{Z} + r\mathbb{Z} = f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])\mathbb{Z}$, where $f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ is a g.c.d. of m and r if $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. By definition, $f(h\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = h \cdot f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ for all $h \in \mathbb{Z}$ and all $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \in M$. Let $t \in \mathbb{N}$ be such that $t \geq 2$. Define the map $f_t : M \to \{0, 1, \dots, t-1\}$ by $f_t(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) \equiv_t f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$. Then $f_t((h_1 t + h_2)\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = f_t(h_2\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right])$ for all $h_1 \in \mathbb{Z}$, all $h_2 \in \{0, 1, \dots, t-1\}$ and all $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] \in M$. Now assume that $t = 25$ and $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} 40 \\ 200 \end{smallmatrix}\right]$. Then $f(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = 40$ and $f(\left[\begin{smallmatrix} m+t \\ r \end{smallmatrix}\right]) = 5$. Hence $f_t(\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]) = 15 \neq 5 = f_t(\left[\begin{smallmatrix} m+t \\ r \end{smallmatrix}\right])$, as desired.

Now we have the following theorem.

Theorem 6.1. *Assume $\mathfrak{g}^\omega = \mathfrak{g}$ if $W_\Pi \cdot \Pi$ is not $A^{(1)}_l$ (see Lemma 5.2). Then $\dim \mathfrak{g}^\omega_\sigma$ with $\sigma \in M \setminus \{0\}$ are listed below.*

(1) *Assume that $W_\Pi \cdot \Pi$ is $X^{(1)}_l$ with $X = A, \dots, G$, and $k(\alpha) = 1$ and $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$, so $L_{\rm sh} = M$, $R_{\rm ex} = \emptyset$, and $L_{\rm lg} = M$ if $R_{\rm lg} \neq \emptyset$ (so $X = B, C, F$ or G). Then we have $\dim \mathfrak{g}^\omega_\sigma = l + 1$ for all $\sigma \in M \setminus \{0\}$.*

(2) *Assume $W_\Pi \cdot \Pi$ is $X^{(1)}_l$ with $X = B, C, F$ or G. Let $r = (\gamma_2, \gamma_2)/(\gamma_1, \gamma_1)$. Assume that $k(\alpha) = (\alpha, \alpha)/(\gamma_1, \gamma_1)$ and $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$, so $L_{\rm sh} = M$, $L_{\rm lg} = \mathbb{Z}\delta \oplus \mathbb{Z}ra$, and $R_{\rm ex} = \emptyset$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in L_{\rm lg} \setminus \{0\}$, and $\dim \mathfrak{g}_{\sigma_2} = |\Pi_{\rm sh}|$ for all $\sigma_2 \in M \setminus L_{\rm lg}$. (This R is isomorphic to $R(\Pi_1, k_1, g_1)$ for which $W_{\Pi_1} \cdot \Pi_1$ is $D^{(2)}_{l+1}$, $A^{(2)}_{2l-1}$, $E^{(2)}_6$ ($l = 4$), or $D^{(3)}_4$ ($l = 2$) respectively, and $k_1(\alpha) = 1$, $g_1(\alpha) = \emptyset$ ($\alpha \in \Pi$).)*

(3) *Assume $W_\Pi \cdot \Pi$ is $D^{(2)}_{l+1}$, $A^{(2)}_{2l-1}$, $E^{(2)}_6$ ($l = 4$), or $D^{(3)}_4$ ($l = 2$). Let $r = (\gamma_2, \gamma_2)/(\gamma_1, \gamma_1)$. Assume that $k(\alpha) = (\alpha, \alpha)/(\gamma_1, \gamma_1)$ and $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$, so $L_{\rm sh} = M$, $L_{\rm lg} = rM$, and $R_{\rm ex} = \emptyset$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in L_{\rm lg} \setminus \{0\}$, and $\dim \mathfrak{g}_{\sigma_2} = |\Pi_{\rm sh}|$ for all $\sigma_2 \in M \setminus rM$.*

(4) *Assume $W_\Pi \cdot \Pi$ is $D^{(2)}_{l+1}$, and $k(\alpha_0) = 2$, $k(\alpha_1) = 1$, $k(\beta) = 2$ ($\beta \in \Pi_{\rm lg}$), $g(\alpha) = \emptyset$ ($\alpha \in \Pi$), so $L_{\rm sh} = \{0, \delta, a\} + M$, $L_{\rm lg} = 2M$, and $R_{\rm ex} = \emptyset$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in 2M \setminus \{0\}$, and $\dim \mathfrak{g}_{\sigma_2} = 1$ for all $\sigma_2 \in M \setminus 2M$.*

(5) *Assume $W_\Pi \cdot \Pi$ is $D^{(2)}_{l+1}$, and $k(\alpha_0) = 2$, $g(\alpha_0) = 2\mathbb{Z} + 1$, $k(\alpha_1) = 1$, $g(\alpha_1) = \emptyset$, $k(\beta) = 2$, $g(\beta) = \emptyset$ ($\beta \in \Pi_{\rm lg}$), so $L_{\rm sh} = \{0, \delta, a\} + M$, $L_{\rm lg} = 2M$ and $\frac{1}{2}L_{\rm ex} = \delta + a + 2M$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in 2M \setminus \{0\}$, and $\dim \mathfrak{g}_{\sigma_2} = 1$ for all $\sigma_2 \in M \setminus 2M$.*

(6) *Assume $W_\Pi \cdot \Pi$ is $D^{(2)}_{l+1}$, and $k(\alpha_0) = 2$, $g(\alpha_0) = 2\mathbb{Z} + 1$, $k(\alpha_1) = 1$, $g(\alpha_1) = 2\mathbb{Z} + 1$, $k(\beta) = 1$, $g(\beta) = \emptyset$ ($\beta \in \Pi_{\rm lg}$), so $L_{\rm sh} = M$, $L_{\rm lg} = \{0, a\} + 2M$, and $L_{\rm ex} = a + 2M$. Then we have $\dim \mathfrak{g}_{\sigma_1} = l + 1$ for all $\sigma_1 \in L_{\rm lg} \setminus \{0\}$, and*

$\dim \mathfrak{g}_{\sigma_2} = 1$ *for all* $\sigma_2 \in M \setminus L_{\mathrm{lg}}$. *(This* R *is isomorphic to* $R(\Pi_2, k_2, g_2)$ *for which* $W_{\Pi_2} \cdot \Pi_2$ *is* $A_{2l}^{(2)}$, *and* $k_2(\alpha) = 1$, $g_2(\alpha) = \emptyset$ $(\alpha \in \Pi_{\mathrm{sh}})$, $k_2(\beta) = 2$, $g_2(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{lg}} \cup \Pi_{\mathrm{ex}})$.*)*

(7) *Assume* $W_\Pi \cdot \Pi$ *is* $A_{2l}^{(2)}$, *and* $k(\alpha) = 1$ $(\alpha \in \Pi)$, $g(\alpha_1) = 2\mathbb{Z}+1$, $g(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{lg}} \cup \Pi_{\mathrm{ex}})$, *so* $L_{\mathrm{sh}} = L_{\mathrm{lg}} = M$, *and* $L_{\mathrm{ex}} = \{\delta, \delta + a, a\} + 2M$. *Then we have* $\dim \mathfrak{g}_\sigma = l+1$ *for all* $\sigma \in M \setminus \{0\}$.

(8) *Assume* $W_\Pi \cdot \Pi$ *is* $B_l^{(1)}$, *and* $k(\alpha) = 1$ $(\alpha \in \Pi)$, $g(\alpha_1) = 2\mathbb{Z}+1$, $g(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{lg}})$, *so* $L_{\mathrm{sh}} = L_{\mathrm{lg}} = M$, *and* $L_{\mathrm{ex}} = a + 2M$. *Let* $M' = \{0, a\} + 2M$. *Then we have* $\dim \mathfrak{g}_{\sigma_1} = l+1$ *for all* $\sigma_1 \in M' \setminus \{0\}$, *and* $\dim \mathfrak{g}_{\sigma_2} = 1$ *for all* $\sigma_2 \in M \setminus M'$. *(This* R *is isomorphic to* $R(\Pi_3, k_3, g_3)$ *for which* $W_{\Pi_3} \cdot \Pi_3$ *is* $A_{2l}^{(2)}$, *and* $k_3(\alpha) = 1$, $g_3(\alpha) = \emptyset$ $(\alpha \in \Pi_{\mathrm{sh}} \cup \Pi_{\mathrm{lg}})$, $k_3(\beta) = 2$, $g_3(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{ex}})$.*)*

(9) *Assume* $W_\Pi \cdot \Pi$ *is* $A_{2l}^{(2)}$, *and* $k(\alpha) = 1$, $g(\alpha) = \emptyset$ $(\alpha \in \Pi)$, *so* $L_{\mathrm{sh}} = L_{\mathrm{lg}} = M$, *and* $L_{\mathrm{ex}} = \{a, \delta + a\} + 2M$. *Then we have* $\dim \mathfrak{g}_{\sigma_1} = l+1$ *for all* $\sigma_1 \in M' \setminus (L_{\mathrm{ex}} \cup \{0\})$, *and* $\dim \mathfrak{g}_{\sigma_2} = 1$ *for all* $\sigma_2 \in L_{\mathrm{ex}}$. *(This* R *is isomorphic to* $R(\Pi_4, k_4, g_4)$ *for which* $W_{\Pi_4} \cdot \Pi_4$ *is* $A_{2l}^{(2)}$, *and* $k_4(\alpha_1) = 1$, $g_4(\alpha_1) = 2\mathbb{Z}+1$, $k_4(\alpha_0) = 2$, $g_4(\alpha_1) = \emptyset$, $k_4(\beta) = 1$, $g_4(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{lg}})$.*)*

(10) *Assume* $W_\Pi \cdot \Pi$ *is* $D_{l+1}^{(2)}$, *and* $k(\alpha) = 1$ $(\alpha \in \Pi)$, $g(\alpha_0) = 2\mathbb{Z}+1$, $g(\beta) = \emptyset$ $(\beta \in \Pi_{\mathrm{lg}} \cup \{\alpha_1\})$. *Then*

$$L_{\mathrm{sh}} = M,\ L_{\mathrm{lg}} = \{0, a\} + 2M \text{ and } L_{\mathrm{ex}} = \{2\delta + a, 2\delta + 3a\} + 4M, \tag{6.1}$$

and we have

$$\dim \mathfrak{g}_{p\delta + za} = \begin{cases} l+1 & \text{if } p \equiv_4 0 \text{ and } \left[\begin{smallmatrix} p \\ z \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \\ 1 & \text{if } p \equiv_2 1, \\ l & \text{if } p \equiv_4 2 \text{ and } z \equiv_2 0, \\ l+1 & \text{if } p \equiv_4 2 \text{ and } z \equiv_2 1. \end{cases} \tag{6.2}$$

(This R *is isomorphic to* $R(\Pi_5, k_5, g_5)$ *for which* $W_{\Pi_5} \cdot \Pi_5$ *is* $A_{2l}^{(2)}$, $k(\alpha) = (\alpha, \alpha)/(\gamma_1, \gamma_1)$, $g_5(\alpha) = \emptyset$ $(\alpha \in \Pi_{\mathrm{lg}})$.*)*

(At this moment, we do not see why $\dim \mathfrak{g}_{p\delta + za}$ are periodic modulo tM for some $t \in \mathbb{N}$. Maybe one of reasons is that $\mathfrak{g}$ may be realized as a 'fixed point' Lie algebra, see also [3], [20].)

Proof. We only prove (10), since (1)-(9) are similarly treated.

Assume $(\alpha_1, \alpha_1) = 1$. Define $\varepsilon_i \in \mathcal{V}$ $(1 \leq i \leq l)$ by $\varepsilon_1 := \alpha_1$ and $\varepsilon_j := \alpha_j + \varepsilon_{j-1}$ $(2 \leq j \leq l)$. Then $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and $\alpha_0 = \delta - \varepsilon_1$. Moreover, we have

$$\begin{aligned} &W_\Pi \cdot \alpha_1 = \cup_{\epsilon \in \{-1,1\}, 1 \leq i \leq l} \epsilon\varepsilon_i + 2\mathbb{Z}\delta, \\ &W_\Pi \cdot \alpha_r = \cup_{\epsilon_1, \epsilon_2 \in \{-1,1\}, 1 \leq i < j \leq l} \epsilon_1\varepsilon_i + \epsilon_2\varepsilon_j + 2\mathbb{Z}\delta \ (2 \leq r \leq l), \\ &W_\Pi \cdot \alpha_0 = \cup_{\epsilon \in \{-1,1\}, 1 \leq i \leq l} \epsilon\varepsilon_i + (2\mathbb{Z}+1)\delta. \end{aligned} \tag{6.3}$$

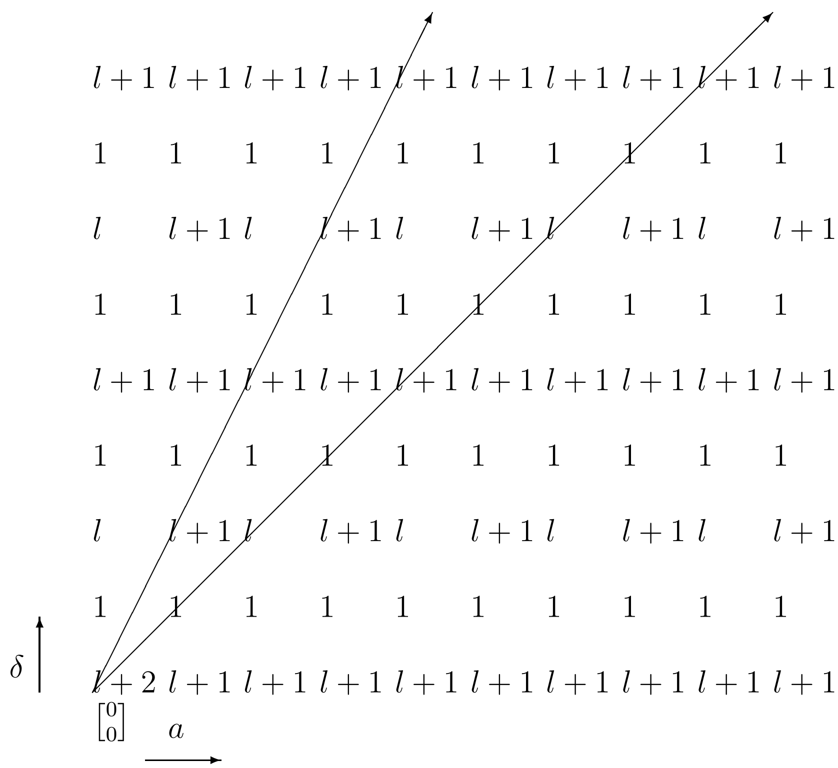


Figure 1: $\dim \mathfrak{g}_{m\delta+ra}$ in (6.2)

Then by (4.9), we have

$$
\begin{aligned}
R &= \cup_{\epsilon\in\{-1,1\},1\le i\le l}\epsilon\varepsilon_i + 2\mathbb{Z}\delta + \mathbb{Z}a \\
&\quad \cup \cup_{\epsilon_1,\epsilon_2\in\{-1,1\},1\le i<j\le l}\ \epsilon_1\varepsilon_i + \epsilon_2\varepsilon_j + 2\mathbb{Z}\delta + \mathbb{Z}2a \\
&\quad \cup \cup_{\epsilon\in\{-1,1\},1\le i\le l}\ \epsilon\varepsilon_i + (2\mathbb{Z}+1)\delta + \mathbb{Z}a \\
&\quad \cup \cup_{\epsilon\in\{-1,1\},1\le i\le l}\ 2(\epsilon\varepsilon_i + (2\mathbb{Z}+1)\delta) + (2\mathbb{Z}+1)a \\
&= \cup_{\epsilon\in\{-1,1\},1\le i\le l}\epsilon\varepsilon_i + M \\
&\quad \cup \cup_{\epsilon_1,\epsilon_2\in\{-1,1\},1\le i<j\le l}\ \epsilon_1\varepsilon_i + \epsilon_2\varepsilon_j + 2M \\
&\quad \cup \cup_{\epsilon\in\{-1,1\},1\le i\le l}\ \epsilon 2\varepsilon_i + (4\mathbb{Z}+2)\delta + (2\mathbb{Z}+1)a.
\end{aligned} \tag{6.4}
$$

Hence we have (6.1), as desired.

Let $\Pi'\cup\{a'\}$ be a fundamental-set of R. Let $\delta' := \delta(\Pi')$, so $\{\delta', a'\}$ is a $\mathbb{Z}$-basis of M.

Assume $a' \equiv_4 a = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Then $\delta' \equiv_4 \delta = \left[\begin{smallmatrix}1\\y\end{smallmatrix}\right]$, where we replace Π' with $-\Pi'$ if necessary. Let $\delta'' = \delta' - ya'$. Then $\{\delta'', a'\}$ is a $\mathbb{Z}$-basis of M. Since $\delta'' \equiv_4 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \equiv_2 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we have $L_{\mathrm{lg}} = \{0, a'\} + 2M$ and $L_{\mathrm{ex}} = \{2\delta''+a', 2\delta''+3a'\} + 4M$. Hence we have the root system isomorphism $f_1 : \mathbb{R}R \to \mathbb{R}R$ (cf. (2.4)) such that $f_1(\alpha_j) = \alpha_j$ $(1 \le j \le l)$, $f_1(\delta) = \delta''$ and $f_1(a) = a'$. Then by Theorem 5.2, we have $\dim \mathfrak{g}_{ma'} = l+1$ for $m \in \mathbb{Z}^\times$.

Assume $a' \equiv_4 \delta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Let $R_5 = R(\Pi_5, k_5, g_5)$ be as in the statement. Let $\mathfrak{g}' := \mathfrak{g}(\Pi_5, k_5, g_5)$. Define the $\mathbb{R}$-linear isometry $f_2 : \mathbb{R}R_5 \to \mathbb{R}R$ by $f_2(\alpha_j) = \alpha_j$ $(1 \le j \le l)$, $f_2(\delta) = 2\delta - a$ and $f_2(a) = \delta$. Note that $f_2(L_{\mathrm{sh}}) = f_2(M) = M = L_{\mathrm{sh}}$, $f_2(L_{\mathrm{lg}}) = f_2(\{0,\delta\}+2M) = L_{\mathrm{lg}}$ and $f_2(L_{\mathrm{ex}}) = f_2(\{\delta, 3\delta\}+4M) = L_{\mathrm{lg}}$. Hence f_2 is a root system isomorphism. Let $a'' := f_2^{-1}(a')$. Then $a'' \equiv_4 a$. By the same argument as above, as for $\dim \mathfrak{g}'_{ma''}$, we have the same equalities as in (6.5) below. Then Theorem 5.2 implies that

$$
\dim \mathfrak{g}_{ma'} = \begin{cases} l+1 & \text{if } m \neq 0 \text{ and } m \equiv_4 0, \\ 1 & \text{if } m \equiv_2 1, \\ l & \text{if } m \equiv_4 2. \end{cases} \tag{6.5}
$$

For other a''s, we can utilize the root system isomorphisms $f_i : \mathbb{Z}R \to \mathbb{Z}R$ $(3 \le i \le 5)$ defined by $f_i(\alpha_j) = \alpha_j$ for all $1 \le j \le l$, and $f_3(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$, $f_3(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $f_4(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $f_4(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$, $f_5(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $f_5(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Let $R_6 = R(\Pi_6, k_6, g_6)$ be such that $W_{\Pi_6}\cdot\Pi_6$ is $D^{(2)}_{l+1}$, $k_6(\alpha_i) = 1$ for $1 \le i \le l$, and $g_6(\alpha_0) = \emptyset$, $g_6(\alpha_1) = 2\mathbb{Z}+1$ and $g_6(\alpha_j) = \emptyset$ for $2 \le j \le l-1$. Then we can also use the root system isomorphism $f_6 : \mathbb{Z}R_6 \to \mathbb{Z}R$ defined by $f_6(\alpha_j) = \alpha_j$ $(1 \le j \le l)$, $f_6(\delta) = \delta$ and $f_6(a) = 2\delta + a$.

Finally we have

Case-1. If $a' \equiv_4 \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]$, then we have $\dim \mathfrak{g}_{ma'} = l+1$ for $m \in \mathbb{Z}^\times$.

Case-2. If $a' \equiv_4 \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]$, then the same as (6.5) holds.

Let $\lambda = p\delta + za = \left[\begin{smallmatrix} p \\ z \end{smallmatrix}\right] = ma'$ with p, $z \in \mathbb{Z}$ and $m \in \mathbb{Z}^\times$. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = a'$, so $x\mathbb{Z} + y\mathbb{Z} = \mathbb{Z}$.

Assume that $p \equiv_4 0$. If $x \equiv_2 1$, then $m \equiv_4 0$, so $\dim \mathfrak{g}_\lambda = l + 1$. If $x \equiv_2 0$, then $y \equiv_2 1$, so Case-1 implies $\dim \mathfrak{g}_\lambda = l + 1$.

Assume that $p \equiv_4 2$ and $z \equiv_2 0$. If $x \equiv_2 0$, then $y \equiv_2 1$, so $m \equiv_2 0$, so $p \equiv_4 0$, contradiction. Hence $x \equiv_2 1$, so $m \equiv_4 2$, so Case-2 implies $\dim \mathfrak{g}_\lambda = l$.

Assume that $p \equiv_4 2$ and $z \equiv_2 1$. Then $m \equiv_2 1$, $y \equiv_2 1$ and $x \equiv_2 0$, so Case-1 implies $\dim \mathfrak{g}_\lambda = l + 1$.

Assume that $p \equiv_2 1$. Then $m \equiv_2 1$ and $x \equiv_2 1$, so Case-2 implies $\dim \mathfrak{g}_\lambda = 1$.

Thus we have (6.2), as desired. This completes the proof. □

Acknowledgment. The authors would like to express their thanks to Kaiming Zhao for a fruitful discussion on Remark 6.1. Most of this paper is motivated by delivery of talks and lectures by the authors at the University of Isfahan, Qom University and Arak University in Iran in September 2005 and December 2008. The authors would like to express their thanks to professors, staffs and students there for hospitality, and careful and warm listening to the authors' presentations.

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